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Basic theoretical concepts and definitions of the lectures in discipline «Higher Mathematics»

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Basic theoretical concepts and definitions of the lectures in discipline «Higher Mathematics» include theoretical justification of the main notions and methods of calculation in the discipline «*Higher Mathematics*»

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Plaskon S.A., Dzyubanovska N.V. Basic theoretical concepts and definitions of the lectures in discipline «Higher Mathematics»**».** - Ternopil: WUNU, 2022.- 44 p.

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1. Elements of Linear algebra

§ 1. Determinants

1. The determinant of the second order is name a number which is after a formula: $\Delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$.

2. The determinant of the third order is name a number which is after a formula:

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{21}a_{32}a_{13} - \\ - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{23}a_{32}a_{11}.$$

3. Determinant n-looks like a go order:

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}.$$

By the minor key M_{ij} element a_{ij} determinant n

order deteminant name is $(n-1)$ - order, got from previous after deletion i -row i j - column.

By addition of algebra A_{ij} element a_{ij} determinant n - the minor key for this element, taken with a sign "+", is named an order, if number $(i+j)$ – pair, and with a sign "-", if it not pair. That

$$\grave{A}_{ij} = (-1)^{i+j} M_{ij}.$$

By a determinant n-number which equals the sum of pair works of elements of arbitrary line (or column) on their proper additions of algebra is named a go order. That

$$\Delta = a_{i1}A_{i1} + a_{i2}A_{i2} + \dots + a_{in}A_{in} \quad (i = 1,2,\dots,n),$$

$$\Delta = a_{1j}A_{1j} + a_{2j}A_{2j} + \dots + a_{nj}A_{nj} \quad (j = 1,2,\dots,n).$$

4.Properties of determinants of random order:

1) does not a determinant change, if to change lines on the proper columns, but columns -on lines;
2) at transposition of two lines (columns) the absolute value of determinant does not change, and a sign changes on opposite;if all elements of some line (column) equal a zero, then a determinant equals a zero also;
3) a determinant which two identical lines (columns) are in equals a zero;
4) if all elements of some line (column) have a general multiplier, then he can be taken away for the sign of determinant;
5) a determinant in which all elements of one line (column) are proportional to the proper elements of other line (column) equals a zero;
6) if all elements of some line (column) of determinant are the sum of two elements, then determinant it is possible to give as a sum of two determinants. Thus are elements of the considered line (column) in the first determinant the first elements, and elements of the proper line (column) of the second determinant - by the second elements;
7) a determinant does not change, if to all elements of arbitrary line (column) to add the elements of other line (column), increased on a the same number;the sum of poparnikh works of elements of some line (column) on additions of algebra of the proper elements of other line (column) equals a zero.

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 Task 1. To calculate determinant of the second orde:

$$\begin{vmatrix} 3 & -4 \\ 2 & 1 \end{vmatrix}.$$

Untiing.

$$\begin{vmatrix} 3 & -4 \\ 2 & 1 \end{vmatrix} = 3 \times 1 - (-4) \times 2 = 3 + 8 = 11.$$

 Task 2. To calculate determinant of the third order

:

$$\begin{vmatrix} 2 & 1 & -3 \\ 3 & 0 & -1 \\ 4 & -2 & 5 \end{vmatrix}.$$

Untiing.

$$\begin{vmatrix} 2 & 1 & -3 \\ 3 & 0 & -1 \\ 4 & -2 & 5 \end{vmatrix} = 2 \times 0 \times 5 + 1 \times (-1) \times 4 + (-3) \times 3 \times (-2) - (-3) \times 0 \times 4 - \\ - 1 \times 3 \times 5 - 2 \times (-1) \times (-2) = 0 - 4 + 18 + 0 - 15 - 4 = \\ = -5.$$

Task 3. To calculate determinant of the third order, decomposing him after the elements of line (or column):

$$\begin{vmatrix} 1 & 2 & 4 \\ 3 & -1 & 0 \\ 1 & 2 & -5 \end{vmatrix}.$$

Untiing.

$$\begin{vmatrix} 1 & 2 & 4 \\ 3 & -1 & 0 \\ 1 & 2 & -5 \end{vmatrix} = 3 \times (-1)^{2+1} \times \begin{vmatrix} 2 & 4 \\ 2 & -5 \end{vmatrix} + (-1) \times (-1)^{2+2} \times \begin{vmatrix} 1 & 4 \\ 1 & -5 \end{vmatrix} +$$

$$+ 0 \times (-1)^{2+3} \begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} = 3 \times (-1)^3 \times (-10 - 8) - 1 \times (-1)^4 \times (-5 - 4) + 0 =$$

$$= -3\times(-18) - 1\times(-9) = 63.$$

Task 4. To calculate determinant of fourth order, using him to property:

$$\begin{vmatrix} 3 & 2 & 1 & 4 \\ 1 & 0 & 1 & 2 \\ -1 & -1 & 3 & 0 \\ 0 & 2 & 1 & 5 \end{vmatrix}.$$

Untiing.

$$\begin{vmatrix} 3 & 2 & 1 & 4 \\ 1 & 0 & 1 & 2 \\ -1 & -1 & 3 & 0 \\ 0 & 2 & 1 & 5 \end{vmatrix} = \begin{vmatrix} 0 & 2 & -2 & -2 \\ 1 & 0 & 1 & 2 \\ 0 & -1 & 4 & 2 \\ 0 & 2 & 1 & -5 \end{vmatrix} = (-1)^3 \begin{vmatrix} 2 & -2 & -2 \\ -1 & 4 & 2 \\ 2 & 1 & -5 \end{vmatrix} =$$

$$= -\begin{vmatrix} 2 & 0 & 0 \\ -1 & 3 & 1 \\ 2 & 3 & -3 \end{vmatrix} = -2\times(-1)^2 \times \begin{vmatrix} 3 & 1 \\ 3 & -3 \end{vmatrix} = -2\times(-9-3) = 24.$$

VECTORIAL ALGEBRA

§ 1. Actions with vectors

1. **A curriculum of vector is after unit vectors:**

$$\vec{u} = X\vec{i} + Y\vec{j} + Z\vec{k}$$ **– in space**

$$\vec{u} = X\vec{i} + Y\vec{j}$$ **–. on a plane**

2. **Projections (co-ordinates)** $X,\ Y,\ Z$ **of vector** $\vec{u}$ **:**

$$\text{np}_x\vec{u} = X = x_2 - x_1,$$
$$\text{np}_y\vec{u} = Y = y_2 - y_1,$$
$$\text{np}_z\vec{u} = Z = z_2 - z_1.$$

3. **Length of vector** $\vec{u}$ **:**

$$|\vec{u}| = \sqrt{X^2 + Y^2 + Z^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

4. Sending cosines:

$$\cos\alpha = \frac{X}{\sqrt{X^2+Y^2+Z^2}}, \qquad \cos\beta = \frac{Y}{\sqrt{X^2+Y^2+Z^2}},$$

$$\cos\gamma = \frac{Z}{\sqrt{X^2+Y^2+Z^2}}.$$

Here $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

5. If $\vec{a}\{X_1,\ Y_1,\ Z_1\}$, $\vec{b}\{X_2, Y_2, Z_2\}$ **then**

$\vec{a} \pm \vec{b} = \vec{c}\{X_1 \pm X_2;\ Y_1 \pm Y_2;\ Z_1 \pm Z_2\}$,

$\lambda\vec{a} = \{\lambda X_1, \lambda Y_1, \lambda Z_1\}$.

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∢ Ex. 1 In a rhombus $ABCD$ diagonals are set $\overrightarrow{AC} = \vec{a}$ and $\overrightarrow{BD} = \vec{b}$. To decompose after these two vectors all of vectors which coincide with the sides of rhombus: $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CD}$, $\overrightarrow{DA}$.

Instruction.

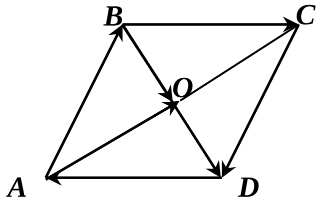
We consider a rhombus $ABCD$.

$$\overrightarrow{AO} = \frac{1}{2}\overrightarrow{AC} = \frac{1}{2}\vec{a};$$

$$\overrightarrow{BO} = \frac{1}{2}\overrightarrow{BD} = \frac{1}{2}\vec{b};$$

$$\overrightarrow{AO} = \overrightarrow{AB} + \overrightarrow{BO};$$

$$\overrightarrow{AB} = \overrightarrow{AO} - \overrightarrow{BO} = \frac{1}{2}(\vec{a} - \vec{b});$$



$$\overrightarrow{CD} = -\overrightarrow{AB} = -\frac{1}{2}(\vec{a} - \vec{b});$$

$$\overrightarrow{BC} = \overrightarrow{AC} - \overrightarrow{AB} = \vec{a} - \frac{1}{2}(\vec{a} - \vec{b}) = \frac{1}{2}(\vec{a} + \vec{b});$$

$$\overrightarrow{DA} = -\overrightarrow{BC} = -\frac{1}{2}(\vec{a} + \vec{b}).$$

∢ Task 2. A vector $\vec{a}$ is set the coordinates of the ends $A(1;3;-2)$ and $B(2;-1;5)$. Please, define coordinates, length and direction of this vector.

Instruction. Find the coordinates of vector $\vec{a}$ as difference between the eventual and initial coordinates of points: $X=2-1=1$**;** ↓ $Y=-1-3=-4$**;** $Z=5-(-2)=7$**.**

Length of vector.

$$|\vec{a}|=\sqrt{X^2+Y^2+Z^2}=\sqrt{1^2+(-4)^2+7^2}=\sqrt{1+16+49}=\sqrt{66}\,.$$

Direction of vector is determined by sending cosines:

$$\cos\alpha=\frac{X}{|a|}=\frac{1}{\sqrt{66}}\,;\quad \cos\beta=\frac{-4}{\sqrt{66}}\,;\quad \cos\gamma=\frac{7}{\sqrt{66}}\,.$$

For verification of our results we'll find:

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma=$$

$$=\left(\frac{1}{\sqrt{66}}\right)^2+\left(-\frac{4}{\sqrt{66}}\right)^2+\left(\frac{7}{\sqrt{66}}\right)^2=\frac{1+16+49}{66}=1.$$

§ 2. Scalar multiplication of two vectors

1. *Scalar multiplication of two vectors* $\vec{a}$ i $\vec{b}$ a number which equals work of lengths of these vectors on the cosine of angle between them is named, that $\vec{a}\cdot\vec{b}=|\vec{a}|\cdot|\vec{b}|\cos\varphi$.

If the known co-ordinates of two vectors $\vec{a}\{X_1;Y_1;Z_1\}$, $\vec{b}\{X_2;Y_2;Z_2\}$, then scalar multiplication of these vectors is evened:

$$\vec{a}\cdot\vec{b}=X_1X_2+Y_1Y_2+Z_1Z_2\,.$$

scalar multiplication thirls:

$$\vec{i}\cdot\vec{j}=\vec{i}\cdot\vec{k}=\vec{j}\cdot\vec{k}=0,\quad \vec{i}\cdot\vec{i}=\vec{j}\cdot\vec{j}=\vec{k}\cdot\vec{k}=1\,.$$

2. A angle is between two vectors:

$$\cos\varphi=\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|}=\frac{X_1X_2+Y_1Y_2+Z_1Z_2}{\sqrt{X_1^2+Y_1^2+Z_1^2}\cdot\sqrt{X_2^2+Y_2^2+Z_2^2}}\,.$$

Condition of perpendicularity of two vectors:

$$X_1X_2+Y_1Y_2+Z_1Z_2=0.$$

Condition of parallelness (to the colinearity) of two vectors $\frac{X_1}{X_2}=\frac{Y_1}{Y_2}=\frac{Z_1}{Z_2}$.

∢ Task 1. To calculate a angle between vectors $\vec{a}\{2;-4;4\}$ **i** $\vec{b}\{-3;2;6\}$.

Instruction .Using a formula $\cos\varphi=\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|}=\frac{X_1X_2+Y_1Y_2+Z_1Z_2}{\sqrt{X_1^2+Y_1^2+Z_1^2}\cdot\sqrt{X_2^2+Y_2^2+Z_2^2}}$**,**

We'll have $\cos\varphi=\frac{2(-3)+(-4)\cdot 2+4\cdot 6}{\sqrt{4+16+16}\cdot\sqrt{9+4+36}}=\frac{5}{21}$**.**

Then $\varphi=\arccos(\frac{5}{21})\approx 76°13'$**.**

§ 3. Vectorial multiplication

1. Vectorial multiplication of two vectors $\vec{a}$ i $\vec{b}$ is named *вектор* $\vec{c}$, that is such characteristics:

a) length of vector $\vec{c}$ equals the area of parallelogram, built on vectors $\vec{a}$ i $\vec{b}$, that

$$|\vec{c}|=|\vec{a}||\vec{b}|\sin(\widehat{\vec{a},\vec{b}});$$

б) vector $\vec{c}$ perpendicular to the plane of this parallelogram, that perpendicular to the vectors $\vec{a}$ i $\vec{b}$;

в) vector $\vec{c}$ directed in such sidewhich the shortest turn is from from $\vec{a}$ to *b* considered carried out against a hour-hand. Vectorial multiplication of two vectors $\vec{a}\{X_1;Y_1;Z_1\}$ **i** $\vec{b}\{X_2;Y_2;Z_2\}$ **get by a determinant:** $\vec{a}\times\vec{b}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ X_1 & Y_1 & Z_1\\ X_2 & Y_2 & Z_2\end{vmatrix}$**.**

2. Area of parallelogram, built on vectors $\vec{a}$ **i** $\vec{b}$:

$$S=|\vec{a}\times\vec{b}|.$$

Area of triangle, built on vectors $\vec{a}$ **i** $\vec{b}$ **:**

$$S_\Delta=\frac{1}{2}|\vec{a}\times\vec{b}|.$$

∢ Task 1. **To find the S of triangle with tops** $A(1;2;0)$, $\boldsymbol{B}$**(3; 0; –3) i** $\boldsymbol{C}$**(5; 2; 6).**

***Instruction:*. S** ABC **equals the half of S of parallelogram, built on vectors** $\overrightarrow{AB}$ **i** $\overrightarrow{AC}$**:**

$$S_\Delta = \frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|.$$

Find vectors $\overrightarrow{AB}$ **i** $\overrightarrow{AC}$**:**

$$\overrightarrow{AB} = 2\vec{i} - 2\vec{j} - 3\vec{k};\quad \overrightarrow{AC} = 4\vec{i} + 6\vec{k}.$$

It's vectorial multiplication will be:

$$\overrightarrow{AB}\times\overrightarrow{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -2 & -3 \\ 4 & 0 & 6 \end{vmatrix} = \vec{i}\begin{vmatrix} -2 & -3 \\ 0 & 6 \end{vmatrix} - \vec{j}\begin{vmatrix} 2 & -3 \\ 4 & 6 \end{vmatrix} + \vec{k}\begin{vmatrix} 2 & -2 \\ 4 & 0 \end{vmatrix} =$$

$= -12\vec{i} - 24\vec{j} + 8\vec{k} = 4(-3\vec{i} - 6\vec{j} + 2\vec{k})$**, so**

$\left|\overrightarrow{AB}\times\overrightarrow{AC}\right| = 4\left|-3\vec{i} - 6\vec{j} + 2\vec{k}\right| = 4\sqrt{(-3)^2 + (-6)^2 + 2^2} = 28$**,**

In the end, $S_\Delta = 14$ **sq.un.**

§ 4. Vectorial spaces

*1***.** Well-organized system *n* of numbers $a_1,\ a_2,\ \ldots,\ a_n$ is named n- *measured vector*. Arbitrary aggregate all *n-measured* vectors for which the set concepts of addition of vectors and increase of vector on a number, named n-vector by the measured vectorial (or linear) space.

Line-dependent vectors are named the vectors $\vec{a}_1,\ \vec{a}_2,\ \ldots,\ \vec{a}_n,$ if there is even one material number $\alpha_i\ (i = 1, 2, \ldots, n),$ that does not equal a zero, and equality is executed:

$$\alpha_1\vec{a}_1 + \alpha_2\vec{a}_2 + \ldots + \alpha_n\vec{a}_n = 0.$$

Line-independent are named the vectors $\vec{a}_1, \vec{a}_2,\ \ldots,\ \vec{a}_n,$ if this equality is executed only then, when all $\alpha_i = 0 \ (i = 1, 2, ..., n)$.

Aggregate of line-independent vectors $\vec{a}_1, \vec{a}_2, \ ..., \ \vec{a}_n,$ at which an arbitrary vector of space is linear combination of vectors of this aggregate, named a base n-measured space.

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∢ Task 1. To define line-dependence or independence of system of vectors $\vec{a}_1\{2; 1; 0\}$, $\vec{a}_2\{0; -2; 1\}$, $\vec{a}_3\{1; 2; -1\}$.

Instruction. **We'll find all values** $\lambda_1, \ \lambda_2, \ \lambda_3,$ **which equality is executed at:** $\lambda_1\vec{a}_1 + \lambda_2\vec{a}_2 + \lambda_3\vec{a}_3 = 0.$

Putting in this equality in place of vectors $\vec{a}_1, \vec{a}_2, \vec{a}_3$ **their coordinates,we'll have:**

$$\lambda_1\vec{a}_1+\lambda_2\vec{a}_2+\lambda_3\vec{a}_3=\lambda_1(2;1;0)+\lambda_2(0;-2;1)+\lambda_3(1;2;-1)=(2\lambda_1;\lambda_1;0)+$$
$$+(0;-2\lambda_2;\lambda_2)+(\lambda_3;2\lambda_3;-\lambda_3)=(2\lambda_1+\lambda_3;\lambda_1-2\lambda_2+2\lambda_3;\lambda_2-\lambda_3).$$

The gotten vector equals a zero vector, so

$$(2\lambda_1 + \lambda_3; \ \lambda_1 - 2\lambda_2 + 2\lambda_3; \ \lambda_2 - \lambda_3) = (0; 0; 0),$$

from here we have:

$$\begin{cases} 2\lambda_1 + \lambda_3 = 0 \\ \lambda_1 - 2\lambda_2 + 2\lambda_3 = 0 \\ \lambda_2 - \lambda_3 = 0 \end{cases}.$$

As an amount of equalizations equals the amount of unknown, and determinant $\begin{vmatrix} 2 & 0 & 1 \\ 1 & -2 & 2 \\ 0 & 1 & -1 \end{vmatrix} = 1 \neq 0$**,**

then the system of linear homogeneous equalizations has a zero decision, that $\lambda_1 = \lambda_2 = \lambda_3 = 0$**. Consequently, set vectors are line-independent.**

Task 2. Four vectors are given $\vec{a}_1\{1; 1; -1\}$, $\vec{a}_2\{1; 2; 1\}$, $\vec{a}_3\{3; 2; 1\}$, $\vec{b}\{1; 7; -1\}$ in some base. To show that vectors $\vec{a}_1, \vec{a}_2, \vec{a}_3$ form a base and to find the coordinates of vector $\vec{b}$ in this base.

Instruction. The Matrix

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ 3 & 2 & 1 \end{bmatrix},$$

made from the coordinates of the set vectors, has a determinant $|A| = 2 - 2 + 3 + 6 - 1 - 2 = 6 \neq 0$, so vectors $\vec{a}_1, \vec{a}_2, \vec{a}_3$ are line-independent and create the base of 3D vectorial space. For determination of coordinates of vector $\vec{b}$ of this base, we have to find such numbers: $\lambda_1, \lambda_2, \lambda_3$, that $\vec{b} = \lambda_1\vec{a}_1 + \lambda_2\vec{a}_2 + \lambda_3\vec{a}_3$ or

$$\begin{bmatrix} 1 \\ 7 \\ -1 \end{bmatrix} = \lambda_1 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \lambda_3 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}.$$

Vectors are equel, when their coordinates are equel. Therefore from the last equality we have $\begin{cases} \lambda_1 + \lambda_2 + 3\lambda_3 = 1 \\ \lambda_1 + 2\lambda_2 + 2\lambda_3 = 7 \\ -\lambda_1 + \lambda_2 + \lambda_3 = -1 \end{cases}$.

We'll find a result of this system of equalizations using the formulas of Cramer. Will find for this purpose:

$$\Delta = \begin{vmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ 3 & 2 & 1 \end{vmatrix} = 6, \qquad \Delta_1 = \begin{vmatrix} 1 & 1 & -1 \\ 7 & 2 & 1 \\ -1 & 2 & 1 \end{vmatrix} = 18,$$

$$\Delta_2 = \begin{vmatrix} 1 & 1 & -1 \\ 1 & 7 & 1 \\ 3 & -1 & 1 \end{vmatrix} = 24, \qquad \Delta_3 = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 7 \\ 3 & 2 & -1 \end{vmatrix} = -12,$$

$$\lambda_1 = \frac{\Delta_1}{\Delta} = \frac{18}{6} = 3, \ \lambda_2 = \frac{\Delta_2}{\Delta} = \frac{24}{6} = 4, \ \lambda_3 = \frac{\Delta_3}{\Delta} = \frac{-12}{6} = -2.$$

So, we have disintegration of vector $\vec{b}$ on the base

$$\vec{b} = 3\vec{a}_1 + 4\vec{a}_2 - 2\vec{a}_3.$$

Coordinates of vector $\vec{b}$ in base $\vec{a}_1, \vec{a}_2, \vec{a}_3$ will be $(3; 4; -2)$.

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UNIT 3. Analytic geometry in the space

§ 1. Plane

1. basic equation of the plane:

$$Ax + By + Cz + D = 0.$$

Here $\vec{N}\{A; B; C\}$ – vector, perpendicular to the plane (normal vector)

2. Normal equation of the plane:

$$x\cos\alpha + y\cos\beta + z\cos\gamma - p = 0.$$

Here $\cos\alpha,\ \cos\beta,\ \cos\gamma$ – gerichtet kosinus of the normal vector.

3. Equation off the plane, which goes through points $M_1(x_1; y_1; z_1)$ and is perpendicular to normal vector $\vec{N}\{A; B; C\}$:

$$A(x - x_1) + B(y - y_1) + C(z - z_1) = 0.$$

4. Equation of the plane in segments on axes:

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$$

Here $a,\ b,\ c$ – segments, which are cut off by plane from datum lines *OX*, *OY*, *OZ*.

5. Eequation of the plane , which goes through three points $M_1(x_1; y_1; z_1),\ M_2(x_2; y_2; z_2),\ M_3(x_3; y_3; z_3)$:

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0.$$

6. Kosinus angle φ, formed by two planes:

$$\cos\varphi = \pm\frac{\vec{N}_1\vec{N}_2}{|\vec{N}_1||\vec{N}_2|} = \frac{A_1A_2 + B_1B_2 + C_1C_2}{\sqrt{A_1^2 + B_1^2 + C_1^2}\cdot\sqrt{A_2^2 + B_2^2 + C_2^2}}.$$

Here $\vec{N}_1$ i $\vec{N}_2$ – normal vectors to the plane

$$A_1x + B_1y + C_1z + D_1 = 0 \text{ and } A_2x + B_2y + C_2z + D_2 = 0.$$

7. Condition when two planes will be parallel:

$$\frac{A_1}{A_2}=\frac{B_1}{B_2}=\frac{C_1}{C_2}.$$

8. Condition when two planes will be perpendicular:

$$A_1A_2+B_1B_2+C_1C_2=0.$$

9. Distance d between point $M_0(x_0;\, y_0;\, z_0)$ and a plane $Ax+By+Cz+D=0$:

$$d=\frac{|Ax_0+By_0+Cz_0+D|}{\sqrt{A^2+B^2+C^2}}.$$

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≺ **Practice 1.** Plane goes through point $P(3;6;-4)$ and separate segments on the axis of absciss $a=-3$ and on the z-axis $c=2$. Write equation of the plane.

Aanswer*:* We have to use $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$. If $a=-3$, $c=2$, then $\frac{x}{-3}+\frac{y}{b}+\frac{z}{2}=1$.

Point P is on the plane, that is why it coordinates satisfy equation of this plane:

$$\frac{3}{-3}+\frac{6}{b}+\frac{-4}{2}=1, \text{ so } b=\frac{3}{2}.$$

Equation of the plane will be $\frac{x}{-3}+\frac{2y}{3}+\frac{z}{2}=1$, or

$$2x-4y-3z+6=0.$$

≺ **Practice 2.**Find distance between point $A(2;3;-1)$ and a plane $7x-6y-6z+42=0$.

Answer. We use this formula for finding distance between point and plane

$$d=\frac{|Ax_0+By_0+Cz_0+D|}{\sqrt{A^2+B^2+C^2}}.$$

Having substituted into the formula values $A=7;\; B=-6;\; C=-6$; $x_0=2;\; y_0=3;\; z_0=-1$, receive

$$d = \left| \frac{7 \cdot 2 + (-6) \cdot 3 + (-6)(-1) + 42}{\sqrt{7^2 + (-6)^2 + (-6)^2}} \right| = \left| \frac{14 - 18 + 6 + 42}{11} \right| = 4 .$$

§ 2. Straight line

1. Canonical equation of the straight line :

$$\frac{x - x_1}{m} = \frac{y - y_1}{n} = \frac{z - z_1}{p} .$$

Here $\vec{S}\{m; n; p\}$ – vector, which is parallel to the straight line (directed vector).

2. Parametrical equation of the straight line:

$$\begin{cases} x = x_1 + mt \\ y = y_1 + nt \\ z = z_1 + pt \end{cases} .$$

3. Equation of the straight line, which goes through two given points $M_1(x_1; y_1; z_1)$, $M_2(x_2; y_2; z_2)$:

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1} .$$

4. Basic equation of the straight line:

$$\begin{cases} A_1 x + B_1 y + C_1 z + D_1 = 0 \\ A_2 x + B_2 y + C_2 z + D_2 = 0 \end{cases} .$$

Here $\vec{N}_1\{A_1; B_1; C_1\}$, $\vec{N}_2\{A_2; B_2; C_2\}$ – normal vectors of two planes.

5. equation of the straight line in projection:

$$x = mz + x_1, \; y = nz + y_1,$$

$$\text{or } \frac{x - x_1}{m} = \frac{y - y_1}{n} = \frac{z - 0}{1} .$$

6. kosinus of the angle φ between two straight lines:

$$\cos\varphi = \frac{m_1 m_2 + n_1 n_2 + p_1 p_2}{\sqrt{m_1^2 + n_1^2 + p_1^2} \cdot \sqrt{m_2^2 + n_2^2 + p_2^2}} .$$

Here $\vec{S}_1\{m_1; n_1; p_1\}$, $\vec{S}_2\{m_2; n_2; p_2\}$ – direct vectors of the straight lines .

7. Two straight lines will be parallel if:

$$\frac{m_1}{m_2}=\frac{n_1}{n_2}=\frac{p_1}{p_2}.$$

8. Two straight lines will be per perpendicular if :

$$m_1m_2+n_1n_2+p_1p_2=0.$$

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∢ **Practice 1.** Write canonical and parametrical equations of the straight line , which goes through point $M_0(1;-2;2)$ and is parallel to the axis OY.

Answer. Vector $\vec{P}\{0;1;0\}$ is on the axis OY and is parallel to the straight line, that is why we can take it as a direct vector of this line. Using formula $\frac{x-x_1}{m}=\frac{y-y_1}{n}=\frac{z-z_1}{p}$ receive canonical and equations of the straight line $\frac{x-1}{0}=\frac{y+2}{1}=\frac{z-2}{0}$. Finding parametrical equations of the straight line we have to remember that

zero in denominator of the first and third relation show , that $x-1=0$ and $z-2=0$.

Equate second relation to the t, and you receive $y=-2+t$.

So, required parametrical equations are $x=1,\ y=-2+t,\ z=2$.

∢ **Practice 2.** Straight line is given by basic equation

$$\begin{cases} x-2y+z=-1 \\ 2x+y-z=3 \end{cases}.$$

Write canonical equations of the straight line.

Answer. Supposing ,that $z=0$. Having solved a system

$$\begin{cases} x-2y=-1 \\ 2x+y=3 \end{cases},$$

We find, that $x=1,\ y=1$, in other words straight line goes through $M_1(1;1;0)$.

If $y=0$, then Having solved

$$\begin{cases} x+z=-1 \\ 2x-z=3 \end{cases},$$

receive, that $x=\frac{2}{3}$, $z=\frac{-5}{3}$, straight line also goes through $M_2\left(\frac{2}{3};0;\frac{-5}{3}\right)$. Using equation of straight line, which goes through two points

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1},$$

We will receive: $\quad \dfrac{x-1}{\frac{2}{3}-1}=\dfrac{y-1}{0-1}=\dfrac{z-0}{-\frac{5}{3}-0}$.

So , canonical equation of the straight line is:

$$\frac{x-1}{-\frac{1}{3}}=\frac{y-1}{-1}=\frac{z}{-\frac{5}{3}}.$$

§ 3. Line and plane

1. equation of the planes which go through the line $\begin{cases} A_1x+B_1y+C_1z+D_1=0 \\ A_2x+B_2y+C_2z+D_2=0 \end{cases}$,

looks like

$$A_1x+B_1y+C_1z+Д_1+k(A_2x+B_2y+C_2z+Д_2)=0,$$

where k – any number.

2. Angle between line and plane:

$$\sin\varphi=\frac{|Am+Bn+Cp|}{\sqrt{A^2+B^2+C^2}\sqrt{m^2+n^2+p^2}}.$$

Here $\vec{N}\{A;B;C\}$ – normal vector of the plane, $\vec{S}\{m;n;p\}$ – direct vector of the line.

3. Line and plane will be parallel if:

$$Am+Bn+Cp=0.$$

4. Line and plane will be perpendicular if:

$$\frac{A}{m}=\frac{B}{n}=\frac{C}{p}.$$

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⮜ **Practice 1.**Find point of the intersection of a line $\frac{x-1}{3} = \ = \frac{y+1}{-1} = \frac{z-2}{5}$ with a plane $x + y - 2z - 4 = 0$.

Answerя. Let write equation of the straight line in parametrical form. Each of relations of which an equation of the line is consist equals t:

$$\frac{x-1}{3} = \frac{y+1}{-1} = \frac{z-2}{5}(=t) \text{ or } \frac{x-1}{3} = t;\ \frac{y+1}{-1} = t;\ \frac{z-2}{5} = t,$$

so $x = 3t+1;\ y = -t-1;\ z = 5t+2.$

As coordinates of the intersection point of the line and a plane have to satisfy equations of the line and plane,we have to substitute values $x,\ y$ and z from parametrical equations of the straight line into equation of the plane, Then we receive

$$3t + 1 + (-t-1) - 2(5t+2) - 4 = 0.$$

Having solved it ,we will find $t = -1$. Value t is value of parameter in the point of the intersection of the line and a plane. Than we have to substitute it in the parametrical equation of the straight line ,and we receive: $x = -2;\ \ y = 0;\ z = -3.$ So, $(-2;\,0;\,3)$ is coordinate of the intersection point of the line and a plane.

⮜ **Practice 2.**Write equation of the plane, which goes through two parallel lines:

$$\frac{x-1}{2} = \frac{y-3}{3} = \frac{z}{4};\ \frac{x+2}{2} = \frac{y+1}{3} = \frac{z-1}{4}.$$

Answer. Writ equation of the first line as :

$$\begin{cases} \frac{x-1}{2} = \frac{y-3}{3} \\ \frac{x-1}{2} = \frac{z}{4} \end{cases},$$

and after simplification

$$\begin{cases} 3x-2y+3=0 \\ 2x-z-2=0 \end{cases}.$$

equations of the planes, which go through this lineare:

$$3x-2y+3+\lambda(2x-z-2)=0$$

or

$$(3+2\lambda)x-2y-\lambda z+3-2\lambda=0.$$

Assign that plane , which goes through the line. Another line goes through point $M(-2;-1;1)$, and that is why to the plane, which goes through another line ,this pint has to belong . Having substituted into an equation coordinates of the point $M(-2;-1;1)$, we receive relation with help of which we find λ.

$$(3+2\lambda)(-2)-2(-1)-\lambda 1+3-2\lambda=0;$$

$$\lambda=-\frac{1}{7}.$$

Having substituted the values λ in the equation, одержимо receive necessary equation of the plane

$$19x-14y+z+23=0.$$

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UNIT 4. **INTRODUCTION TO MATHEMATICAL ANALYSIS**

§ 1. Notion of the function

Aggregate of all material numbers which satisfy $a<x<b$, inequality named an interval and reflected named an interval and reflected (a,b) or $]a,b[$.

Aggregate of all material numbers which satisfy inequality $a\le x\le b$, named a segment and reflected $[a,b]$.

2.The absolute value $|a|$ The absolute value a of material number a is name a number a if a positive or equals a zero and number a if subzero, that

$$|a| = \begin{cases} a, & \text{якщо} \quad a \ge 0, \\ -a, & \text{якщо} \quad a < 0. \end{cases}$$

4.A variable quantity y is named the function of variable quantity x (reflected y=f(x)), if a law after which to every value x to taken from an area possible values is indicated, answers an actual value is certain the Variable quantity is named an independent variable or argument y. x ($y = f(x)$),

The range of definition of function y=f(x) is name an aggregate all those values of argument x, which values exist for, that is certain material numbers. Range values of function y=f(x) the aggregate of all values is x named when x changes in the range of definition of this function

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∢ **Task 1.**To define, what values x inequality is executed at $|x-3|<2.$

Soluting. The set inequality can be written down so: $-2 < x - 3 < 2$. To every part of this inequality will add for 3 and obsessed $-2+3<x<2+3,$ that $1<x<5.$ Consequently, inequality $|x-3|<2$ executed for all values x from an interval $(1,5)$.

∢ **Task 2.**To find the range of definition of function $y = \sqrt{2-x}$.

Untiing.In order that a function y had actual values only, size $2-x,$ that is under a root, must not have subzero values, but must be $2-x \ge 0,$ that $x \le 2.$ The range of definition of function is an aggregate of actual values x that less or evened 2, that $x \in (-\infty; 2]$.

§ 2. Notion of function

1. A number a is named graniceyu of numerical sequence x_1, x_2, …, x_n, …, if for arbitrary as pleasingly small positive number ε there will be such natural number $N,$ that at $n \ge N$ inequality is executed

$$|x_n - a| < \varepsilon .$$

In this case write down: $\lim\limits_{n\to\infty} x_n = a$.

2. Number b named verge of function $f(x)$ that $x \to a$, if for an arbitrary numerical sequence $x_n \to a \ (x_n \neq a)$ values of argumenta x the proper sequence

3. $y_1 = f(x_1),\ y_2 = f(x_2),\ ...,\ y_n = f(x_n), \dots$

values of function y has graniceyu a number b.

Write down like $\lim\limits_{x \to a} f(x) = b.$

4. If exist $\lim\limits_{x\to a} f(x) = A$ i $\lim\limits_{x\to a} g(x) = B$, то:

1) $\lim\limits_{x\to a}(f(x) \pm g(x)) = \lim\limits_{x\to a} f(x) \pm \lim\limits_{x\to a} g(x) = A \pm B$;

2) $\lim\limits_{x\to a} f(x)\cdot g(x) = \lim\limits_{x\to a} f(x) \cdot \lim\limits_{x\to a} g(x) = A \cdot B$;

3) $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)} = \dfrac{A}{B} \quad \left(\lim\limits_{x\to a} g(x) = B \neq 0\right).$

5. First prominent verge:

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

6. Second prominent verge:

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \lim_{\alpha\to 0}(1+\alpha)^{\frac{1}{\alpha}} = e \approx 2{,}72.$$

7. At untiing of tasks use formulast:

$$\lim_{x\to\infty}\left(1+\frac{k}{x}\right)^x = e^k, \quad \lim_{\alpha\to 0}(1+k\alpha)^{\frac{1}{\alpha}} = e^k.$$

8. Formulas of growth are after compound interests:

$$K_t = K_0\left(1+\frac{p}{100}\right)^t = K_0(1+i)^t,$$

where K_t – sum of deposit, accumulated through t years; K_0 – initial sum of deposit; p – year-on-year percent increase; t – a period of growth is in years; $i = {}^{p}\!/_{100}$, $1+i=r$ – coefficient of compound interests. Continuous growth is after compound interests:

$$K_t = K_0 \cdot e^{\frac{p}{100}t} = K_0 \cdot e^{it}$$

If $p>0$, a formula is named the pokaznikovim law of growth, and at $p<0$ – by the law of slump. Eventual size K_t initial sum K_0 t years in the case when specific interest rate – i, and percents are counted m one times per a year, calculate on a formula:

$$K_t = K_0 \left(1+\frac{i}{m}\right)^{mt}.$$

10. Accounts of accumulation:

$$S = P \cdot S_{n/i}$$

This S - size of account of accumulation;

P - initial payment;

$S_{n/i}$ - it is in the calculation table of D8 (in additions).

11. Calculations of rent:

$$A = P \cdot a_{n/i},$$

де $a_{n/i} = i^{-1}\left[1-(1+i)^{-n}\right]$ tabbed for different values $i = \frac{R}{100}$ and n.

This A- a size of payment is on the rent account;

P - annual payment by n years;

R - size of year-on-year percent growth.

12.Debt retirement:

$$P = \frac{A}{a_{n/i}}.$$

This A -size of payment taken in a debt;

P - sum of the regular returning;

$a_{n/i}$ - it is in the calculation table of D8 (in additions).

∢ **Task 1.**To find $\lim\limits_{x\to 2}\frac{\sqrt{x^2+5}-3}{x-2}$.

Untiing. At $x \to 2$ a numerator and denominator of shot have verge which equals a zero. Will carry irrationality in a denominator, increasing a numerator and denominator on conjugating expression to the numerator, that on, that on

$\sqrt{x^2+5}+3$, obsessed:

$$\lim_{x\to 2}\frac{\sqrt{x^2+5}-3}{x-2}=\lim_{x\to 2}\frac{(\sqrt{x^2+5}-3)(\sqrt{x^2+5}+3)}{(x-2)(\sqrt{x^2+5}+3)}=$$

$$=\lim_{x\to 2}\frac{x^2+5-9}{(x-2)(\sqrt{x^2+5}+3)}=\lim_{x\to 2}\frac{x^2-4}{(x-2)(\sqrt{x^2+5}+3)}=$$

$$=\lim_{x\to 2}\frac{(x-2)(x+2)}{(x-2)(\sqrt{x^2+5}+3)}=\lim_{x\to 2}\frac{(x+2)}{(\sqrt{x^2+5}+3)}=\frac{2+2}{3+3}=\frac{2}{3}.$$

∢ **Task 2.**In town lives 249 thousands of inhabitants. Annually a population is increased on 1,7%. What amount of habitants will be in this town in 12 years?

*Untiing.*Will use the formula of growth after compound

interests: $K_{12}=249\left(1+\frac{1,7}{100}\right)^{12}\approx 305$.

Consequently, in 12 years there will be 305 thousands of habitants in town.

∢ **Task 3.**A depositor gives a bank 2000 hryvnyas under compound interests with a condition them continuous growth on 12% annual. To calculate piling up of capital for 4 years.

*Untiing.*Will use the formula of neperevnogo growth after compound interests:

$$K_4=2000\cdot e^{4\cdot 0,12}\approx 3,2322 \text{ тис. грн.}$$

∢ **Задача 4.** Sum $K_0=200$ inlaid under compound interests from a calculation 12% annual a term on 4 years. To calculate an eventual sum, if percents are counted at the end of every month.

*Untiing.*It is known that $K_0=200$ тис. грн., $i=0,12,\ m=12,\ t=4$. Consequently,

$$К_4=200\left(1+\frac{0,12}{12}\right)^{12\cdot 4}=200\cdot 1,01^{48}=322,4 \text{ тис. грн.}$$

∢ **Task 5.** Each month a student places 100 hryvnyas to the account of accumulation with the receipt of income 5% monthly. To calculate the size of his accumulation after realization of a 12 payment.

Untiing. As a tabular value $S_{n/i}$ even $S_{n/i} = S_{12/0,05} = 15{,}917127$, that $S = 100 \cdot 15{,}917127 \approx 1591{,}71$ uah..

∢ **Task 6.** In the day of the 55-richchya workwoman of firm "Oster" scored first rent in an insurance company "UNIVERSAL" on condition of annual receipt in the birthday 1000 Uah during 15. What sum is placed to the account of rent, if a money is accepted with a 5% year-on-year growth?

Untiing. Will use a formula $A = P \cdot a_{n/i}$. In our task regular payments $P = 1000$ uah.. Coefficient $a_{n/i}$ it is taken from the table of D8 and even $a_{15/0,05} = 10{,}379658$. Means $A = 1000 \cdot 10{,}379658 \approx 10379{,}66$ uah

Consequently, the workwoman of firm must lay 10379,66 Uah on the account of rent, to get for 1000 Uah annually during 15.

∢ **Task 7.** In a time of studies the student of university got from a fund studies in a debt 8000 Uah This credit it is given him from a 8% year-on-year growth the condition of the annual returning of debt at the end of every year upon termination of university during 15. How many money must a student return every year upon termination of university?

Untiing. The sought after size of P of annual satisfaction of debt a student is after a formula $P = \dfrac{A}{a_{n/i}}$.

In this case debt A=8000 of Uah, time of his returning $n = 15$, percent of growth $R = 8$, $i = \dfrac{R}{100} = 0{,}08$. From the table of D 8 find $a_{15/0,08} = 8{,}559479$. Thays why

$$P = \frac{8000}{a_{15/0,08}} = \frac{8000}{8{,}559479} \approx 934{,}64 \text{ грн..}$$

Consequently, for debt retirement a student must at the end of every year pay the fund of studies 934,64 Uah

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§ 3. The concept of continuity of a function

1.Increase of argument and function:

$\Delta x = x_2 - x_1,$
$\Delta y = y_2 - y_1 = f(x_1 + \Delta x) - f(x_1).$

Function $y = f(x)$ named continuous in a point x_0, якщо нескінченно малому приросту аргументу Δx $x = x_0$ a small increase answers infinitely Δy to the function which is certain in a point x_0 . That at $\Delta x \to 0$ буде $\Delta y \to 0$.

Function $y = f(x)$ named continuous at $x = x_0$,if :

1) $f(x)$ it is existed at $x = x_0$ and in some points x_0;
2) there is left-side verge $\lim\limits_{x \to x_0 - 0} f(x);$
3) there is right-side verge $\lim\limits_{x \to x_0 + 0} f(x);$

4)left-side and right-side granici is levels $\lim\limits_{x \to x_0 - 0} f(x) = \lim\limits_{x \to x_0 + 0} f(x);$

$\lim\limits_{x \to x_0} f(x) = f(x_0)$ regardless of method of sent x to x_0, that regardless of method of sent $\lim\limits_{x \to x_0 - 0} f(x) = \lim\limits_{x \to x_0 + 0} f(x) = f(x_0).$

1) Classification of points of breaks of function
2) If function $f(x)$ not certain in a point x_0 or certain, but correlations take place

$$\lim_{x \to x_0 - 0} f(x) = \lim_{x \to x_0 + 0} f(x) \neq f(x_0),$$

3) break likvidovnim is named in a point . If one-sided verge of function $\lim\limits_{x \to x_0 - 0} f(x),$ $\lim\limits_{x \to x_0 + 0} f(x)$ exist completion, but not even between itself x_0 exist completion, but not even between itself $\Delta = \lim\limits_{x \to x_0 + 0} f(x) - \lim\limits_{x \to x_0 - 0} f(x)$ named the jump of function. If though one of one-sided verges does not exist or evened ∞, break in this point named the break of the second family. Such breaks of the first and second family are named nelikvidovnimi

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∢ **Task 1.** To find the interval of continuity of function

$$f(x) = 4x^2 - 3x + 6 .$$

*Untiing.*Will take an arbitrary point on numerical wasp and will designate through Δx increase of argument x Then the set function will get an increase

$$\Delta y = f(x_0 + \Delta x) - f(x_0) = \left[4(x_0 + \Delta x)^2 - 3(x_0 + \Delta x) + 6\right] -$$
$$-(4x_0^2 - 3x_0 + 6) = 4x_0^2 + 8x_0\Delta x + 4(\Delta x)^2 - 3x_0 - 3\Delta x + 6 -$$
$$-4x_0^2 + 3x_0 - 6 = 4(\Delta x)^2 + (8x_0 - 3)\Delta x.$$

Now find verge Δy if $\Delta x \to 0$:

$$\lim_{\Delta x\to 0} \Delta y = \lim_{\Delta x\to 0}\left[4(\Delta x)^2 + (8x_0 - 3)\Delta x\right] = 0.$$

Consequently, a function is continuous in a point x_0. This assertion takes place for any point of numerical axis, that is why function $f(x) = 4x^2 - 3x + 6$ continuous on all numerical wasp.

∢ **Task 2.** To probe on continuity and find the points of break of function $f(x) = \dfrac{4}{x-2}$.

Untiing. The set function is continuous for all values *x* except for *x=2.* As a denominator *x-2* of shot equals a zero at *x=2* function $f(x)$ bursting at $x = 2$. Will define character of this point of break. Will find left-side granicyu of function at first $\lim\limits_{x\to 2-0} f(x)$. When $x \to 2-0$, it is possible to put $x = 2 - \alpha$ $(\alpha>0)$ and to consider that α, remaining dodatneyu, heads for a zero:

Now will define right-side verge of function. If $x \to 2+0$, we can change $x = 2 + \alpha$ $(\alpha>0)$ and to consider that α, remaining positive, heads for a zero. Replacing x on $2+\alpha$, we have:

$$\lim_{x\to 2+0} f(x) = \lim_{x\to 2+0}\frac{4}{x-2} = \lim_{\alpha\to 0}\frac{4}{2+\alpha-2} = \lim_{\alpha\to 0}\frac{4}{\alpha} = +\infty .$$

Thus, here are neither granici on the left nor granici business, and that is why a point of x=2 is a point of break of the second family.

∢ **Task 3.** The bureau of economic analysis of VAT “Vatra” set that at the production of x units of products of An every quarter charges are expressed a formula V(x)

$$V(x) = 20\,000 + 40x \text{ (uah)},$$

And profit $D(x)$, got from the sale of x units of this products expressed a formula

$$D(x) = 100x - 0{,}001x^2 \text{ (uah)}.$$

Each quarter a factory is produced by 3100 units of products And, but aims to increase the issue of this products to 3200 units. To calculate the increase of charges, profit and income. To find the average of increase of income on unit of increase of *products.*

Untiing.The planned increase of products will be Δx = 3200 – 3100 = 100 (unit of products A).

Increase of charges:

$$\Delta V(x) = V(3200) - V(3100) = (20000 + 40 \cdot 3200) - -(20000 + 40 \cdot 3100) = 148000 - 144000 = 4000.$$

Increase_of_profit: $\Delta D(x) = D(3200) - D(3100) = (100 \cdot 3200 - 0{,}01 \cdot 3200^2) - -(100 \cdot 3100 - 0{,}01 \cdot 3100^2) = 217600 - 213900 = 3700.$

Will designate the income of *P(x).* Then

$$P(x) = D(x) - V(x) = 100x - 0{,}01x^2 - (20000 + 40x) =$$

$$= -20000 + 60x - 0{,}01x^2.$$

An increase of income will be:

$\Delta P(x) = P(3200) - P(100) = (-20000 + 60 \cdot 3200 - 0{,}01 \cdot 3200^2) -$
$-(-20000 + 60 \cdot 3100 - 0{,}01 \cdot 3100^2) = 69600 - 69900 = -300,$ that will diminish on 300 hryvnyas. An average of income will be on unit of increase of products

$$\frac{\Delta P(x)}{\Delta(x)} = \frac{-300}{100} = -3.$$

Consequently, every unit of additional products diminishes an income on 3 of unit.

UNIT 5. ***DIFFERENTIAL CALCULATION OF FUNCTION OF ONE***

VARIABLE

§ 1 . Differentiation of functions

1. ***Derivate of function*** $y = f(x)$ in point x is named borderline relation to the increase of function Δy to the increase of argument Δx, when Δx arbitrarily heads for a zero. If this granicya exists, it is reflected: y', or $f'(x)$, or y'_x, або $\frac{dy}{dx}$, or $\frac{df(x)}{dx}$.

Mathematically the derivative of function is determined after a formula:

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}.$$

2. 2. Geometrical maintenance of derivative: derivative $f'(x)$ equals the angular coefficient of tangent to the chart of function $y = f(x)$ in a point with an abscissa x.

Mechanical maintenance of derivative: derivative $S'(t)$ it is the size of instantaneous speed in moment t body which moves after a law $S = S(t)$.

Economic maintenance of derivative: derivative $K'(x)$ equals the maximum charges of production $K = K(x)$ homogeneous products as function of amount of products x.

3. If function $y = f(x)$ has a derivative $(n-1)$ to the order, differentiated in some point of interval $[a;b]$, derivative from $f^{(n-1)}(x)$ named a th derivative, or and mark the derivative of th order: $f^{(n)}(x)$, or $y^{(n)}$, or $y_x^{(n)}$, or $\frac{d^n y}{dx^n}$, or $\frac{d^n f(x)}{dx^n}$.

Therefore, -na the derivative of function is determined equality:

$$y^{(n)} = [y^{(n-1)}]'.$$

4. ***Basic formulas of differentiation:***:

1) $y = u(x) \pm v(x)$, $\qquad y' = u' \pm v'$;

2) $y=u(x)\cdot v(x)$, $y'=u'v+uv'$;

3) $y=c,\ c=const$, $y'=0$;

4) $y=c\cdot u(x)$, $y'=cu'$;

5) $y=\dfrac{u(x)}{v(x)}$, $y'=\dfrac{u'v-uv'}{v^2}$;

6) $y=f(u), u=\varphi(x), y=f(\varphi(x))$, $y'=y'_u\cdot u'_x$;

7) $y=u^{\alpha}(x)$, $y'=\alpha\cdot u^{\alpha-1}\cdot u'$;

8) $y=\sqrt{u(x),}$ $y'=\dfrac{u'}{2\sqrt{u}}$;

9) $y=\sin u(x)$, $y'=\cos u\cdot u'$;

10) $y=\cos u(x)$, $y'=-\sin u\cdot u'$;

11) $y=\operatorname{tg} u(x)$, $y'=\dfrac{u'}{\cos^2 u}$;

12) $y=\operatorname{ctg} u(x)$, $y'=-\dfrac{u'}{\sin^2 u}$;

13) $y=a^{u(x)}$, $y'=a^u\cdot\ln a\cdot u'$;

14) $y=e^{u(x)}$, $y'=e^u\cdot u'$;

15) $y=\log_a u(x)$, $y'=\dfrac{u'}{u}\log_a e=\dfrac{u'}{u\ln a}$;

16) $y=\ln u(x)$, $y'=\dfrac{u'}{u}$;

17) $y=\arcsin u(x)$, $y'=\dfrac{u'}{\sqrt{1-u^2}}$;

18) $y=\arccos u(x)$, $y'=-\dfrac{u'}{\sqrt{1-u^2}}$;

19) $y=\operatorname{arctg} u(x)$, $y'=\dfrac{u'}{1+u^2}$;

20) $y=\operatorname{arcctg} u(x)$, $y'=-\dfrac{u'}{1+u^2}$.

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∢ **Task 1.** To find the derivative of function $y=3x^2$ at $x=4$.

Untiing. *Will find the decision of this task, going out from determination. If argument* x gets an increase Δx, то для функції $y=f(x)=3x^2$ will find an increase Δy, that

$$\begin{aligned}
&f(x+\Delta x)=3(x+\Delta x)^2=3x^2+6x\Delta x+3(\Delta x)^2,\\
&\Delta y=f(x+\Delta x)-f(x)=3x^2+6x\Delta x+3(\Delta x)^2-3x^2=\\
&=6x\Delta x+3(\Delta x)^2=(6x+3\Delta x)\Delta x.
\end{aligned}$$

Will divide the increase of function Δy on the increase of argument Δx, that will find middle speed of change of the set function $y=3x^2$ on an interval $(x, x+\Delta x)$.

For finding of derivative y' it is needed to find granicyu of the got relation at $\Delta x\to 0$ (here x it is considered a permanent size). Thus

$$y'=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\frac{(6x+3\Delta x)\Delta x}{\Delta x}=\lim_{\Delta x\to 0}(6x+3\Delta x)=6x.$$

At $x=4$ value of derivative $y'(4)=6\cdot 4=24$. It a number 24 is speed of change of function $y=3x^2$ at $x=4$.

∢ **Task 2.** **To find the derivatives of functions**:

а) $y=\sqrt{x^2+1}+\sqrt[3]{x^3+1}$; б) $y=\ln\sqrt{\dfrac{1-\sin x}{1+\sin x}}$.

Untiing.

a) Will use the rule of differentiation for the sum of two differentiated functions, and then will find the derivatives of difficult functions:

$$y'=(\sqrt{x^2+1}+\sqrt[3]{x^3+1})'=(\sqrt{x^2+1})'+(\sqrt[3]{x^3+1})'=\left((x^2+1)^{\frac{1}{2}}\right)'+$$

$$+\left((x^3+1)^{\frac{1}{3}}\right)'=\frac{1}{2}(x^2+1)^{-\frac{1}{2}}(x^2+1)'+\frac{1}{3}(x^3+1)^{-\frac{2}{3}}(x^3+1)'=$$

$$=\frac{1}{2\sqrt{x^2+1}}\cdot 2x+\frac{1}{3\sqrt[3]{(x^3+1)^2}}\cdot 3x^2=\frac{x}{\sqrt{x^2+1}}+\frac{x^2}{\sqrt[3]{(x^3+1)^2}}.$$

b) Calculate the natural logarithm of function, and then will find the derivative of difficult function:

$$y=\ln\sqrt{\frac{1-\sin x}{1+\sin x}}=\ln\left(\frac{1-\sin x}{1+\sin x}\right)^{\frac{1}{2}}=\frac{1}{2}\ln(1-\sin x)-\frac{1}{2}\ln(1+\sin x);$$

$$y'=\left(\frac{1}{2}\ln(1-\sin x)-\frac{1}{2}\ln(1+\sin x)\right)'=\left(\frac{1}{2}\ln(1-\sin x)\right)'-$$

$$-\left(\frac{1}{2}\ln(1+\sin x)\right)'=\frac{1}{2}\frac{(1-\sin x)'}{1-\sin x}-\frac{1}{2}\frac{(1+\sin x)'}{1+\sin x}=\frac{1}{2}\frac{-\cos x}{1-\sin x}-$$

$$-\frac{1}{2}\frac{\cos x}{1+\sin x}=\frac{-2\cos x}{2(1-\sin^2 x)}=-\frac{1}{\cos x}.$$

∢ **Task 3.** To find the derivative of the third order of function $y=\sin^2 x$.

Untiing. Will find the derivative of the first order, as a derivative of function of degree:

$$y'=(\sin^2 x)'=\left[(\sin x)^2\right]'=2\sin x(\sin x)'=2\sin x\cdot\cos x=\sin 2x.$$

Find the derivative of the second order as a derivative from the found result for y', that $y''=(y')'$. like $y'''=(y'')'$. Consequently,

$$y''=(\sin 2x)'=\cos 2x(2x)'=2\cos 2x;$$
$$y'''=(2\cos 2x)'=2(-\sin 2x)(2x)'=-4\sin 2x.$$

∢ **Task 4.** To find a derivative y' non-obvious function

$$x^2+y^2=9.$$

Untiing. In the set equalization a function is found ambiguously, that is why it is named non-obvious. Differentiating both parts of equality, obsessed $2x+2yy'=0$. From here have $yy'=-x$. Deciding this equalization relatively y', find, that $y'=-\dfrac{x}{y}$.

Here at differentiation of second addition $\left[(y^2)_x'=2yy'\right]$ the derivative of function of degree is at first found, and basis is later differentiated y пo to the independent variable x (that y').

UNIT 6. **INDEFINITE INTEGRAL**

§ 1Method of direct integration

1. Function $F(x)$ named a primitive function for set function $f(x)$, if for arbitrary x from an area determination $f(x)$ equality is executed $F'(x)=f(x)$ or $dF(x)=f(x)dx.$

2. Aggregate of all primitive $F(x)+C$ for the set function $f(x)$ named an indefinite integral and reflected $\int f(x)dx.$ consequently $\int f(x)dx=F(x)+C.$

Table of basic integrals:

1) $\int u^{\alpha}du=\frac{u^{\alpha+1}}{\alpha+1}+C(\alpha\neq-1)$;

2) $\int\frac{du}{u}=\ln|u|+C$;

3) $\int a^{u}du=\frac{a^{u}}{\ln a}+C$;

4) $\int\sin u du=-\cos u+C$;

5) $\int\cos u du=\sin u+C$;

6) $\int\frac{du}{\sin^{2}u}=-\mathrm{ctg}\,u+C$;

7) $\int\frac{du}{\cos^{2}u}=\mathrm{tg}\,u+C$;

8) $\int\frac{du}{\sqrt{a^{2}-u^{2}}}=\arcsin\frac{u}{a}+C,\ \ (-\arccos\frac{u}{a}+C)$;

9) $\int\frac{du}{\sqrt{u^{2}\pm a^{2}}}=\ln\left|u+\sqrt{u^{2}\pm a^{2}}\right|+C$;

10) $\int\frac{du}{a^{2}+u^{2}}=\frac{1}{a}\mathrm{arctg}\frac{u}{a}+C,\ \ (-\frac{1}{a}\mathrm{arcctg}\frac{u}{a}+C)$;

11) $\int\frac{du}{a^{2}-u^{2}}=\frac{1}{2a}\ln\left|\frac{a+u}{a-u}\right|+C$;

12) $\int\frac{du}{u^{2}-a^{2}}=\frac{1}{2a}\ln\left|\frac{u-a}{u+a}\right|+C$;

13) $\int e^u du = e^u + C$;

14) $\int \mathrm{tg}\, u du = -\ln|\cos u| + C$;

15) $\int \mathrm{ctg}\, u du = \ln|\sin u| + C$;

16) $d\int f(u)du = f(u)du$;

17) $\int dF(u) = F(u) + C$;

18) $\int kf(u)du = k\int f(u)du$;

19) $\int [f_1(u) \pm f_2(u)]du = \int f_1(u)du \pm \int f_2(u)du$.

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Task 1. To find an indefinite integral

$$\int (4x^3 + \frac{1}{2\sqrt{x}} - 5\sqrt[3]{x^2})dx.$$

Untiing. Using formulas (1), (16) i (17) tables, obsessed:

$$\int (4x^3 + \frac{1}{2\sqrt{x}} - 5\sqrt[3]{x^2})dx = \int 4x^3 dx + \int \frac{1}{2\sqrt{x}} dx - \int 5\sqrt[3]{x^2} dx = 4\int x^3 dx +$$

$$+\frac{1}{2}\int x^{-\frac{1}{2}} dx - 5\int x^{\frac{2}{3}} dx = 4\cdot\frac{x^4}{4} + \frac{1}{2}\cdot\frac{x^{\frac{1}{2}}}{\frac{1}{2}} - 5\cdot\frac{x^{\frac{5}{3}}}{\frac{5}{3}} + C = x^4 + \sqrt{x} - 3x\sqrt[3]{x^2} + C.$$

∢ **Task 2. To find an indefinite integral** $\int \frac{xdx}{x^2-5}$.

Untiing. As $xdx = \frac{1}{2}d(x^2-5)$, bringing an integral over to to tabular, have

$$\int \frac{xdx}{x^2-5} = \frac{1}{2}\int \frac{d(x^2-5)}{x^2-5} = = \frac{1}{2}\ln\left|x^2-5\right| + C.$$

UNIT 7. CERTAIN INTEGRAL

§ 1. Calculation of certain integrals

1. A certain integral from a function $f(x)$ on a segment $[a;b]$ is name granicya of integral sum on condition that length most from the elementary cuttings-off heads for a zero:

$$\int_a^b f(x)dx = \lim_{\max \Delta x_i \to 0} \sum f(\alpha_i)\Delta x_i.$$

If a function $f(x)$ is continuous on $[a;b]$, granitsya of integral sum exists and does not depend on the method of division of segment $[a;b]$, to pieces and from the choice of points α_i

2. Basic properties of certain integral:

2.1. $\int_a^b f(x)dx = -\int_b^a f(x)dx$.

2.2. $\int_a^a f(x)dx = 0$.

2.3. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx, \quad a < c < b$.

2.4. $\int_a^b [f_1(x) \pm f_2(x)]dx = \int_a^b f_1(x)dx \pm \int_a^b f_2(x)dx$.

2.5. $\int_a^b Cf(x)dx = C\int_a^b f(x)dx$.

2.6. $\int_a^b f(x)dx = f(c)(b-a), \quad a < c < b$.

2.7. Even if m and M least and most value of function $f(x)$ on a segment $[a; b]$, then: $m(b-a) \le \int_a^b f(x)dx \le M(b-a)$.

2.8. If $f(x) \le \varphi(x)$, тthan $\int_a^b f(x)dx \le \int_a^b \varphi(x)dx, \ x \in [a; b]$.

3. Rule calculation of certain integrals:

3.1. Formula of Newton - Leybnica:

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a),$$

where $F(x)$ primitive for $f(x)$, that $F'(x) = f(x)$.

3.2. Replacement of variable:

$$\int_a^b f(x)dx = \int_\alpha^\beta f[\varphi(t)]\varphi'(t)dt,$$

where $x = \varphi(t)$ function, continuous together with the derivative $\varphi'(t)$ on a segment $\alpha \le t \le \beta$, $a = \varphi(\alpha)$, $b = \varphi(\beta)$, $f[\varphi(t)]$ a function is continuous on $[\alpha; \beta]$.

3.3. Integration parts:

$$\int_a^b u dv = u \cdot v\Big|_a^b - \int_a^b v du,$$

3.4. If $f(x)$ continuous function, that $f(-x) = -f(x)$, then:

$$\int_{-a}^{a} f(x)dx = 0$$

If $f(x)$ odd function, that $f(-x) = f(x)$, then

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$$

Task 1. To calculate a certain integral $\int_{\frac{1}{2}}^{\frac{\sqrt{2}}{2}} \frac{4dx}{\sqrt{1-x^2}}$.

Untiing. Using property 2.5, will find primitive to the function:

$$\int_{\frac{1}{2}}^{\frac{\sqrt{2}}{2}} \frac{4dx}{\sqrt{1-x^2}} = 4\int_{\frac{1}{2}}^{\frac{\sqrt{2}}{2}} \frac{dx}{\sqrt{1-x^2}} = 4\arcsin x \Big|_{\frac{1}{2}}^{\frac{\sqrt{2}}{2}} = 4(\arcsin\frac{\sqrt{2}}{2} -$$

$$-\arcsin\frac{1}{2}) = 4\left(\frac{\pi}{4} - \frac{\pi}{6}\right) = \frac{\pi}{3}.$$

Task 2. To calculate a certain integral

$$\int_{0}^{\frac{\pi}{2}} \cos^3 x \sin x dx.$$

Untiing. Will do replacement of variable, will put $t = \cos x.$ Then $dt = -\sin x dx,$ and $\sin x dx = -dt.$ will Find the new limits of integration:

If $x = 0$, then $t = \cos 0 = 1;$

If $x = \frac{\pi}{2}$, then $t = \cos\frac{\pi}{2} = 0$

By such rank

$$\int_0^{\frac{\pi}{2}} \cos^3 x \sin x dx = -\int_1^0 t^3 dt = -\frac{1}{4}t^4\Big|_1^0 = -\frac{1}{4}(0^4 - 1^4) = \frac{1}{4}.$$

Task 3. To calculate a certain integral $\int_0^1 xe^{-x}dx.$

Untiing. Will use the method of integration parts. Will put $u = x,\ dv = e^{-x}dx$, then

$$du = dx,\ v = \int e^{-x}dx = -\int e^{-x}dx = -e^{-x}.$$

$$\int_0^1 xe^{-x}dx = -xe^{-x}\Big|_0^1 - \left(-\int_0^1 e^{-x}dx\right) = -xe^{-x}\Big|_0^1 + \int_0^1 e^{-x}dx =$$

$$= -e^{-1} - e^{-x}\Big|_0^1 = -2e^{-1} + 1 = \frac{e-2}{e}.$$

SECTION 8. **INDEFINITE INTEGRAL**

§ 1Method of direct integration

1. Function $F(x)$ named a primitive function for set function $f(x)$, if for arbitrary x from an area determination $f(x)$ equality is executed $F'(x) = f(x)$ or $dF(x) = f(x)dx.$

2. Aggregate of all primitive $F(x) + C$ for the set function $f(x)$ named an indefinite integral and reflected $\int f(x)dx.$ consequently $\int f(x)dx = F(x) + C.$

Table of basic integrals:

20) $\int u^{\alpha} du = \dfrac{u^{\alpha+1}}{\alpha+1} + C (\alpha \neq -1)$;

21) $\int \dfrac{du}{u} = \ln|u| + C$;

22) $\int a^{u} du = \dfrac{a^{u}}{\ln a} + C$;

23) $\int \sin u du = -\cos u + C$;

24) $\int \cos u du = \sin u + C$;

25) $\int \dfrac{du}{\sin^2 u} = -\mathrm{ctg}\, u + C$;

26) $\int \dfrac{du}{\cos^2 u} = \mathrm{tg}\, u + C$;

27) $\int \dfrac{du}{\sqrt{a^2 - u^2}} = \arcsin\dfrac{u}{a} + C, \quad (-\arccos\dfrac{u}{a} + C)$;

28) $\int \dfrac{du}{\sqrt{u^2 \pm a^2}} = \ln\left|u + \sqrt{u^2 \pm a^2}\right| + C$;

29) $\int \dfrac{du}{a^2 + u^2} = \dfrac{1}{a}\mathrm{arctg}\dfrac{u}{a} + C, \quad (-\dfrac{1}{a}\mathrm{arcctg}\dfrac{u}{a} + C)$;

30) $\int \dfrac{du}{a^2 - u^2} = \dfrac{1}{2a}\ln\left|\dfrac{a+u}{a-u}\right| + C$;

31) $\int \dfrac{du}{u^2 - a^2} = \dfrac{1}{2a}\ln\left|\dfrac{u-a}{u+a}\right| + C$;

32) $\int e^{u} du = e^{u} + C$;

33) $\int \mathrm{tg}\, u du = -\ln|\cos u| + C$;

34) $\int \mathrm{ctg}\, u du = \ln|\sin u| + C$;

35) $d\int f(u) du = f(u) du$;

36) $\int dF(u) = F(u) + C$;

37) $\int kf(u) du = k\int f(u) du$;

38) $\int [f_1(u) \pm f_2(u)]du = \int f_1(u)du \pm \int f_2(u)du$.

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Task 1. To find an indefinite integral

$$\int (4x^3 + \frac{1}{2\sqrt{x}} - 5\sqrt[3]{x^2})dx.$$

Untiing. Using formulas (1), (16) i (17) tables, obsessed:

$$\int (4x^3 + \frac{1}{2\sqrt{x}} - 5\sqrt[3]{x^2})dx = \int 4x^3dx + \int \frac{1}{2\sqrt{x}}dx - \int 5\sqrt[3]{x^2}\,dx = 4\int x^3dx +$$

$$+\frac{1}{2}\int x^{-\frac{1}{2}}dx - 5\int x^{\frac{2}{3}}dx = 4\cdot\frac{x^4}{4} + \frac{1}{2}\cdot\frac{x^{\frac{1}{2}}}{\frac{1}{2}} - 5\cdot\frac{x^{\frac{5}{3}}}{\frac{5}{3}} + C = x^4 + \sqrt{x} - 3x\sqrt[3]{x^2} + C.$$

∢ **Task 2. To find an indefinite integral** $\int \frac{xdx}{x^2-5}$.

Untiing. As $xdx = \frac{1}{2}d(x^2-5)$, bringing an integral over to to tabular, have $\int \frac{xdx}{x^2-5} = \frac{1}{2}\int \frac{d(x^2-5)}{x^2-5} = = \frac{1}{2}\ln\left|x^2-5\right| + C.$

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§ 2. Method of substitution (replacement of variable)

1. Integral $\int f(x)dx$ calculated by a substitution $x = \varphi(t)$, that equality takes place:

$$\int f(x)dx = \int f[\varphi(t)]\varphi'(t)dt.$$

Here $x = \varphi(t)$ – differentiated function of new variable t has reverse $t = \varphi(x)$.

2. At replacement $t = \varphi(x)$ get a formula:

$$\int f[\varphi(x)]\varphi'(x)dx = \int f(t)dt.$$

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∢ **Task 1. To find an indefinite integral** $\int \frac{x^2dx}{\sqrt{9-x^2}}$.

Untiing. Will do a substitution $x = 3\sin t$, then $dx = 3\cos t dt$, $t = \arcsin\frac{x}{3}$, $\sqrt{9-x^2} = \sqrt{9-9\sin^2 t} = \sqrt{9(1-\sin^2 t)} = 3\cos t.$

Therefore

$$\int \frac{x^2dx}{\sqrt{9-x^2}} = \int \frac{9\sin^2 t \cdot 3\cos t dt}{3\cos t} = 9\int \sin^2 t dt = \frac{9}{2}\int (1-\cos 2t)dt =$$

$$= \frac{9}{2}\int dt - \frac{9}{2}\int \cos 2t dt = \frac{9}{2}t - \frac{9}{4}\sin 2t + C =$$

$$= \frac{9}{2}\arcsin\frac{x}{3} - \frac{9}{4}\cdot\frac{2x}{9}\sqrt{9-x^2} + C = \frac{9}{2}\arcsin\frac{x}{3} - \frac{x}{2}\sqrt{9-x^2} + C.$$

Transformation is here done:

$$\sin 2t = 2\sin t\cos t = 2\sin t\sqrt{1-\sin^2 t} = 2\cdot\frac{x}{3}\sqrt{1-\frac{x^2}{3}} = \frac{2}{9}x\sqrt{9-x^2}.$$

Task 2. To find an indefinite integral

$$\int \frac{(2\ln x+1)^4}{x}dx.$$

Untiing. In a subintegral function one part is the function of degree with basis $2\ln x + 1$, and the second part is a differential of this function within a permanent multiplier. Therefore will do a substitution: $2\ln x + 1 = t$, тоді $2\frac{dx}{x} = dt$, $\frac{dx}{x} = \frac{1}{2}dt.$

Consequently,

$$\int \frac{(2\ln x+1)^4}{x}dx = \int (2\ln x+1)^4\cdot\frac{dx}{x} = \int t^4\cdot\frac{1}{2}dt = \frac{1}{2}\int t^4 dt =$$

$$= \frac{1}{2}\cdot\frac{t^5}{5} + C = \frac{1}{10}(2\ln x+1)^5 + C.$$

∢ Task 3. To find an indefinite integral $\int\frac{e^{2x}dx}{e^x+1}$.

Untiing. Will do a substitution: $e^x = t,\ x = \ln t,\ dx = \frac{dt}{t}$. Therefore

$$\int\frac{e^{2x}dx}{e^x+1} = \int\frac{t^2\cdot\frac{dt}{t}}{t+1} = \int\frac{t}{t+1}dt = \int\frac{t+1-1}{t+1}dt = \int\left(1-\frac{1}{t+1}\right)dt =$$

$$= \int dt - \int\frac{dt}{t+1} = t - \ln|t+1| + C = e^x - \ln\left|e^x+1\right| + C.$$

DIFFERENTIAL EQUATIONS

§ 1. Differential equations with the separated variables

1. Ordinary differential equalization is name such equalization, which links an independent variable, its function and derivative (or differentials) of this function. This determination can be written down so:

$$F(x, y, y', y'', ..., y^{(n)}) = 0.$$

The general decision of differential equalization of the order is name a function $y,$ that depends on an argument x and independent arbitrary permanent $C_1, C_2, ..., C_n$ that looks like $y = \varphi(x, C_1, C_2, ..., C_n),$ and which at its substitution in equalization converts him into an identity.

The general integral of differential equalization is name such his general decision which is not untied relatively that $\Phi(x, y, C_1, C_2, ..., C_n) = 0.$

2. Ordinary differential equalization of the first order looks like:

$$F(x, y, y') = 0,$$

or in untied relatively derivative y':

$$y' = f(x, y).$$

The general decision of differential equalization of the first order is name such function $y = \varphi(x, C),$ what at the arbitrary value of permanent is the decision of the set equalization, that at a substitution in equalization in place of unknown function converts him into an identity.

Decision, got from a general decision $y = \varphi(x, C)$ at the defined value of permanent named partial

Task of finding of partial decision which satisfies initial conditions $y = y_0,\ x = x_0$ named a task Koshi.

This task looks like

$$\begin{cases} F(x, y, y') = 0 \\ y = y_0 \ \text{при} \ x = x_0. \end{cases}$$

Built on a plane XOY graph of every equalization $y = \varphi(x)$ set differential equalization named the integral curve of this equalization.

3. Differential equalization of the first order of kind

$$f(x)dx + \varphi(y)dy = 0$$

named equalization with the separated variables

4. Differential equalization of the first order of kind

$$f_1(x)\cdot\varphi_1(y)dx + f_2(x)\cdot\varphi_2(y)dy = 0$$

named equalization with the separated variables.

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Task 1.To find the general decision of differential

$$\frac{1}{x}dx + \frac{1}{y}dy = 0.$$

Untiing. In the set equalization at dx and at dy functions are written down that depend only on x and y accordingly. That is why this equalization with the separated variables, and will find his general decision by integration:

$$\int\frac{dx}{x} + \int\frac{dy}{y} = 0,\ \ln|x| + \ln|y| = \ln C.$$

Here permanent more comfortable to designate integration through $\ln C.$

$$\ln|yx| = \ln C,\quad yx = C.$$

A general decision will be $y = \dfrac{C}{x}$

Task 2. To find the general integral of differential equalization

$y\sqrt{1+x^2}\cdot y' + x\cdot\sqrt{1+y^2} = 0,$

and also partial integral which satisfies initial conditions

$y = 0$ at $x = \sqrt{3}.$

Untiing. Will write down this differential equalization in such kind:

$$y\sqrt{1+x^2}\cdot\frac{dy}{dx} + x\sqrt{1+y^2} = 0.$$

Will increase both parts of equalization on dx:

$$y\sqrt{1+x^2}\cdot dy + x\cdot\sqrt{1+y^2}dx = 0.$$

Will bring this equalization over with the separated variables to equalization with the separated variables by the division of him on $\sqrt{1+x^2}\cdot\sqrt{1+y^2}$:

$$\frac{x}{\sqrt{1+x^2}}dx + \frac{y}{\sqrt{1+y^2}}dy = 0.$$

Integrating both parts of this equalization obsessed

$$\int\frac{x}{\sqrt{1+x^2}}dx + \int\frac{y}{\sqrt{1+y^2}}dy = C.$$

From here the general integral of this equalization will be

$$\sqrt{1+x^2}+\sqrt{1+y^2}=C.$$

Will get a partial integral from a condition $y=0$ at $x=\sqrt{3}.$ Putting these values x and y in the found general integral, obsessed

$$\sqrt{1+3}+\sqrt{1+0}=C,\ C=3.$$

A partial integral will be:

$$\sqrt{1+x^2}+\sqrt{1+y^2}=3.$$