DOI 10.1007/s10958-015-2446-x

Journal of Mathematical Sciences, Vol. 208, No. 3, July, 2015

ESTIMATES OF THE RATE OF POINTWISE AND UNIFORM CONVERGENCE FOR ONE-PERIODIC BRANCHED CONTINUED FRACTIONS OF A SPECIAL FORM

D. I. Bodnar and M. M. Bubnyak

UDC 517.52

A new formula for the difference between two approximants of one-periodic branched continued fractions of a special form is constructed. An estimate for the rate of pointwise and uniform convergence of fractions of this sort is obtained with the help of this formula.

A continued fraction

$$1+\mathop{\mathrm{D}}_{n=1}^{\infty}\frac{a_n}{1} = 1+\frac{a_1}{1} \ + \ \frac{a_2}{1} \ + \ \ddots\,, \tag{1}$$

for which the sequence of elements a_n is k-periodic ($a_{kn+p} = a_p$ for all $n \in \mathbb{N}$, $1 \le p < k$), is called k-periodic, $k \in \mathbb{N}$. The 1-periodic fractions were studied by L. Euler and D. Bernoulli. The k-periodic fractions were considered by E. Kahl, E. Galois, W. Leighton, O. Perron, R. Lane, H. Wall, W. Jones, W. Thron, H. Waadeland, L. Lorentzen, et al. The surveys of the corresponding results can be found in [3–7, 9].

We now consider a linear-fractional mapping $t(\omega)=\dfrac{a\omega+b}{c\omega+d}$, where $a, b, c, d, \omega \in \mathbb{C}$ are such that $ab-dc \ne 0$, $c \ne 0$. Assume that the mapping $t(\omega)$ has two fixed points x and y. The point x is called attracting if $t^n(\omega) = t(t^{n-1}(\omega)) \to x$ as $n \to \infty$ for all $\omega \ne y$. Then y is a repelling point of this mapping.

For a 1-periodic continued fraction

$$1+\frac{c_1}{1} \ + \ \frac{c_1}{1} \ + \ldots\,, \tag{2}$$

where $c_1 \in \mathbb{C}$, we have

$$t(\omega) = 1+\frac{c_1}{\omega}, \qquad x = \frac{1+\sqrt{1+4c_1}}{2}, \qquad y = \frac{1-\sqrt{1+4c_1}}{2}. \tag{3}$$

It is worth noting that, here and in what follows, we take the principal branch $(\sqrt{1}=1)$ of the square roots.

It is known [5–9] that fraction (2) is convergent in the region

$$G = \left\{ z \in \mathbb{C} : \left| \arg\left(z+\frac{1}{4} \right) \right| < \pi \right\}, \tag{4}$$

Ternopil' National Economic University, Ternopil', Ukraine.

Translated from Matematychni Metody ta Fizyko-Mekhanichni Polya, Vol. 56, No. 4, pp. 24–32, October–December, 2013. Original article submitted February 11, 2013.

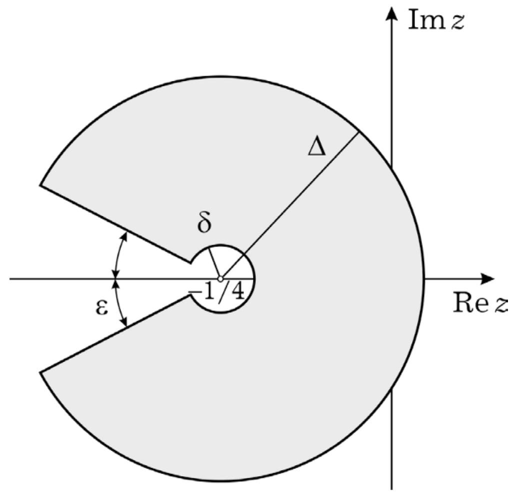

Fig. 1

$x \neq y$, $x \neq 0$, and its n th approximant is equal to

$$f_n = 1 + \mathop{\mathrm{D}}_{k=1}^{n} \frac{c_1}{1} = \frac{x^{n+2} - y^{n+2}}{x^{n+1} - y^{n+1}} = x \frac{1 - \left(\frac{y}{x}\right)^{n+2}}{1 - \left(\frac{y}{x}\right)^{n+1}}, \qquad n \geq 0, \qquad f_0 = 1. \tag{5}$$

Lemma 1. *Let the element* c_1 *of fraction (2) belong to a closed domain*

$$D = \left\{ z \in \mathbb{C} : \left| \arg\left(z + \frac{1}{4} \right) \right| \leq \pi - \varepsilon, \delta \leq \left| z + \frac{1}{4} \right| \leq \Delta \right\}, \tag{6}$$

where $0 < \varepsilon < \pi$, $0 < \delta < \Delta$ *(Fig. 1). Then:*

(i) fraction (2) uniformly converges in the domain D *to the attracting point* x *given by relation (3);*

(ii) the following estimate of the rate of convergence is true:

$$|f_n - x| \leq L\rho^{n+1}, \qquad n \geq 0, \tag{7}$$

$$\rho = \sqrt{\frac{1 - 4\sqrt{d}\sin\frac{\varepsilon}{2} + 4d}{1 + 4\sqrt{d}\sin\frac{\varepsilon}{2} + 4d}}. \tag{8}$$

Moreover, $d = \delta$ *for* $\delta\Delta \leq \frac{1}{16}$ *and* $d = \Delta$ *for* $\delta\Delta > \frac{1}{16}$, $L = \frac{2\sqrt{\Delta}}{1-\rho}$.

Proof. Since $D \subset G$, relation (5) implies that

$$|f_n - x| = \left|\frac{x^{n+2} - y^{n+2}}{x^{n+1} - y^{n+1}} - x\right| = \left|\frac{y}{x}\right|^{n+1} \frac{|x-y|}{\left|1-\left(\frac{y}{x}\right)^{n+1}\right|} \le \left|\frac{1-\sqrt{1+4c_1}}{1+\sqrt{1+4c_1}}\right|^{n+1} \frac{|x-y|}{1-\left|\frac{y}{x}\right|}.$$

If $c_1 \in D$, then the maximum value of $\left|\frac{1-\sqrt{1+4c_1}}{1+\sqrt{1+4c_1}}\right|$ is equal to ρ, which can be found from relation (8). The uniform convergence follows from estimate (7). Lemma 1 is proved.

Fraction (1) is called limiting periodic if the sequence of its elements satisfies the condition $\lim\limits_{n\to\infty} a_n = a^*$.

Parallel with limiting periodic fractions, we also consider the finite fractions

$$h_n = 1 + \frac{a_n}{1} + \frac{a_{n-1}}{1} + \ldots + \frac{a_1}{1}, \qquad n \ge 1, \qquad h_0 = 1, \tag{9}$$

In the terminology of [6, p. 48], they are called reversed fractions.

Lorentzen, Waadeland, and Thron studied the convergence of limiting periodic fractions and fractions of the form (9) in [5, 6, 8].

Lemma 2. *Let* $\{h_n\}_{n=0}^{\infty}$ *be a sequence of reversed fractions of the form (9) whose elements satisfy the condition* $|a_n| \le |a| < \frac{1}{4}$, $n \ge 1$. *Then the following inequalities hold:*

$$|h_n| > \xi, \qquad \xi = \frac{1+\sqrt{1-4|a|}}{2}, \qquad n \ge 0. \tag{10}$$

The proof follows from the fact that the minimant of the fraction h_n is the n th approximant of (2), where $c = -|a|$. The sequence of approximants of the minimant $\{g_n\}_{n=0}^{\infty}$ monotonically decreases and $\lim\limits_{n\to\infty} g_n = \xi$, where ξ is the attracting point of the linear-fractional mapping $t(\omega) = 1 + \frac{-|a|}{\omega}$. Hence, $|h_n| > \xi$.

Consider a 1-periodic branched continued fraction

$$\left(1 + \mathop{\mathrm{D}}_{k=1}^{\infty} \sum_{i_k=1}^{i_{k-1}} \frac{c_{i_k}}{1}\right)^{-1}, \tag{11}$$

where $c_j \in \mathbb{C}$, $j = 1, \ldots, N$, and $i_0 = N$ is an integer.

The finite branched continued fractions

$$F_n = \left(1 + \mathop{\mathrm{D}}_{k=1}^{n} \sum_{i_k=1}^{i_{k-1}} \frac{c_{i_k}}{1}\right)^{-1}, \qquad n \ge 1, \qquad F_0 = 1,$$

are called the n th approximants of the branched continued fraction (11) and the quantities

$$R_n^{(q)} = 1 + \mathop{\mathrm{D}}_{k=1}^{n} \sum_{j_k=1}^{j_{k-1}} \frac{c_{j_k}}{1}, \qquad 1 \le q \le N, \qquad n \ge 1,$$

are called the n th tails of the q th order for fraction (11) ($j_0 = q$, $R_0^{(q)} = 1$, $R_n^{(0)} = 1$).

If $R_n^{(q)} \ne 0$, $q = 1,\ldots,N$, $n \ge 1$, then the following recurrence relations are true:

$$R_n^{(q)} = 1 + \sum_{j=1}^{q} \frac{c_j}{R_{n-1}^{(q)}} = R_n^{(q-1)} + \frac{c_q}{R_{n-1}^{(q)}}, \tag{12$'$}$$

$$R_{n+m}^{(q)} - R_n^{(q)} = R_{n+m}^{(q-1)} - R_n^{(q-1)} + \frac{(-1)c_q}{R_{n+m-1}^{(q)} R_{n-1}^{(q)}} (R_{n+m-1}^{(q)} - R_{n-1}^{(q)}). \tag{12$''$}$$

Let

$$I_{n+1}^{(q)} = \left\{ k = (k_1, k_2, \ldots, k_q) : k_\ell \ge 0,\ \ell = 1,\ldots,q,\ \sum_{\ell=1}^{q} k_\ell = n+1 \right\},$$

$$I_{n+1}^{(j,q)} = \{ k \in I_{n+1}^{(q)} : k_i \ge 1,\ i = 1,\ldots,j,\ k_{j+1} = \ldots = k_q = 0 \}, \qquad q = 1,\ldots,N, \qquad j = 1,\ldots,q,$$

be the sets of multiindices. Then

$$I_{n+1}^{(q)} = \bigcup_{j=1}^{q} I_{n+1}^{(j,q)},$$

$$I_{n+1}^{(q+1,q+1)} = \bigcup_{s=1}^{n+1} (I_{n+1-s}^{(q)} \times \{s\}), \tag{13}$$

and $I_{n+1}^{(k,q)} \cap I_{n+1}^{(\ell,q)} = \varnothing$ if $k \ne \ell$.

We now deduce the formula for the difference of two approximants of the branched continued fraction (11), by using a scheme proposed in [2, p. 28]. In this case, it is assumed that the product in which the superscript is smaller than the subscript is equal to 1.

Lemma 3. *The difference of two approximants of the branched continued fraction (11) is given by the formula*

$$F_{n+m} - F_n = \frac{(-1)^{n+1}}{R_{n+m}^{(N)} R_n^{(N)}} \sum_{k \in I_{n+1}^{(N)}} \frac{c_1^{k_1} c_2^{k_2} \ldots c_N^{k_N}}{\prod_{j=1}^{N} \prod_{r=1}^{k_j} (R_{p_j+m-r}^{(j)} \hat{R}_{p_j-r}^{(j)})}, \tag{14}$$

where $n \geq 0$, $m \geq 1$, $p_j = n - \sum_{\ell=j+1}^{N} k_\ell$, and

$$\hat{R}_n^{(q)} = \begin{cases} R_n^{(q)}, & n \geq 0, \\ 1, & n = -1. \end{cases}$$

Proof. By induction on q, for fixed $n \geq 0$ and $m \geq 1$, we prove the equality

$$R_{n+m}^{(q)} - R_n^{(q)} = (-1)^n \sum_{k \in I_{n+1}^{(q)}} \frac{c_1^{k_1} c_2^{k_2} \ldots c_q^{k_q}}{\prod_{j=1}^{q} \prod_{r=1}^{k_j} (R_{p_j+m-r}^{(j)} \hat{R}_{p_j-r}^{(j)})}. \tag{15}$$

For $q = 1$, equality (15) follows from relations (12′) and (12″) with $p_1 = n$:

$$R_{n+m}^{(1)} - R_n^{(1)} = \frac{(-1)^n c_1^{n+1}}{\prod_{r=1}^{n+1} R_{n+m-r}^{(1)} \hat{R}_{n-r}^{(1)}}.$$

Assume that equality (15) holds for $q = s$. After elementary transformations, in view of relations (12) and (13), for $q = s+1$, we get

$$R_{n+m}^{(s+1)} - R_n^{(s+1)} = R_{n+m}^{(s)} - R_n^{(s)} + \sum_{p=1}^{n} \frac{(-1)^p c_{s+1}^p}{\prod_{s=1}^{p} R_{n+m-s}^{(s+1)} R_{n-s}^{(s+1)}} (R_{n+m-p}^{(s)} - R_{n-p}^{(s)})$$

$$= (-1)^n \sum_{k \in I_{n+1}^{(s+1)}} \frac{c_1^{k_1} c_2^{k_2} \ldots c_{s+1}^{k_{s+1}}}{\prod_{j=1}^{s+1} \prod_{r=1}^{k_j} (R_{p_j+m-r}^{(j)} \hat{R}_{p_j-r}^{(j)})}.$$

In view of the relation $F_{n+m} - F_n = \frac{1}{R_{n+m}^{(N)}} - \frac{1}{R_m^{(N)}}$, the proof of the lemma is completed.

We now construct the domains Ω_j for the choice of elements c_j, $j = 1, \ldots, N$, of fraction (11).

Let $\Omega_1 = G$, where the domain G is defined by relation (4). We choose an element $c_1 \in \Omega_1$ and fix it. Denote

$$p_1 = \left| \frac{y_1}{x_1} \right| = \left| \frac{1 - \sqrt{1 + 4c_1}}{1 + \sqrt{1 + 4c_1}} \right|,$$

where x_1 and y_1 are the fixed points of the linear-fractional mapping (3).

We now choose and fix an element c_2 from the region

$$\Omega_2 = \left\{ z \in \mathbb{C} : |z| < \frac{r_1}{4} \right\}, \qquad r_1 = |x_1|^2 \frac{1-p_1^3}{1+p_1}. \tag{16}$$

Let the elements $c_j \in \Omega_j$, $j = 3, \ldots, k$, be chosen and fixed. Then the region Ω_{k+1} is denoted as follows:

$$\Omega_{k+1} = \left\{ z \in \mathbb{C} : |z| < \frac{1}{4} \prod_{j=1}^{k} r_j \right\}, \qquad r_j = \xi_j^2, \qquad \xi_j = \frac{1+d_j}{2}, \qquad k \le N-1, \tag{17}$$

$$d_j = \sqrt{1 - 4|c_j| \prod_{k=1}^{j-1} r_k^{-1}}. \tag{18}$$

Theorem 1. *Let the elements of fraction (11) belong to the regions constructed above, i.e.,* $c_j \in \Omega_j$, $j = 1, \ldots, N$.

Then

(i) fraction (11) converges;

(ii) the following estimate of the rate of pointwise convergence is true:

$$|F_n - F| \le C_{n+N-1}^{N-1} L p^{n+1}, \qquad n \ge 1. \tag{19}$$

Here, $p = \max_{j=1,\ldots,N} \{p_j\}$, $p_j = \dfrac{1}{(1+d_j)^2}$, *and* d_j, $j = 2, \ldots, N$, *are given by relations (18),*

$$L = \frac{4^N}{\cos^2 \alpha} M_1 \prod_{j=2}^{N} \frac{M_j}{(1+d_j)^2}, \qquad 2\alpha = \begin{cases} \arg c_1, & \arg c_1 \ne \pi, \\ 0, & \arg c_1 = \pi, \end{cases}$$

$$M_1 = \frac{|x_1|(1+p_1)}{(1-p_1)^2}, \qquad M_j = \frac{|c_j| 4^j}{p_j \cos^2 \alpha \prod_{m=2}^{j} (1+d_m)^2}, \qquad j = 2, \ldots, N;$$

(iii) $F = \left(\prod_{j=1}^{N} x_j \right)^{-1}$ *is the value of fraction (11), where*

$$x_j = \frac{1}{2} \left(1 + \sqrt{1 + 4 c_j \prod_{p=1}^{j-1} x_p^{-2}} \right).$$

Proof. By induction on q, we prove that $\left|R_n^{(q)}\right| \ge K_q$, where

$$K_q = \frac{\cos\alpha}{2}\prod_{j=2}^{q}\frac{1+d_j}{2}, \qquad n \ge 0, \qquad q = 1,\ldots,N.$$

Note that $R_n^{(1)} = f_n$ is the n th approximant of the 1-periodic fraction (2). Therefore, for any $c_1 \in \Omega_1$, we get $c_1 \in P(\alpha)$, where

$$P(\alpha) = \left\{ z \in \mathbb{C} : |z| - \operatorname{Re}(ze^{-2i\alpha}) \le \frac{1}{2}\cos^2\alpha \right\}$$

and α is defined in item (ii) of the theorem. The parabolic theorem 3.43 [6, p. 151] implies that $R_n^{(1)} \in V(\alpha)$, where

$$V(\alpha) = \left\{ z \in \mathbb{C} : \operatorname{Re}(ze^{-i\alpha}) \ge \frac{1}{2}\cos\alpha \right\}.$$

Since $\operatorname{dist}(0;\partial V(\alpha)) = K_1$, we obtain $\left|R_n^{(1)}\right| \ge K_1$, $n \ge 0$. By using relation (5), we also establish the estimate $\left|R_n^{(1)} R_{n-1}^{(1)}\right| \ge r_1$.

Under the assumption that $\left|R_n^{(s)}\right| \ge K_s$, $n \ge 0$, $2 \le s \le q$, we prove that the inequalities $\left|R_n^{(q+1)}\right| \ge K_{q+1}$, $n \ge 0$, hold. Note that all $R_n^{(s)} \ne 0$, $n \ge 0$, $2 \le s \le q$. In view of of Proposition 1 in [1, p. 9], we find $R_n^{(q+1)} = R_n^{(q)} h_n^{(q+1)}$, $n \ge 0$, where $h_n^{(q+1)}$ is a reversed fraction of the form

$$h_n^{(q+1)} = 1 + \frac{\dfrac{c_{q+1}}{R_n^{(q)}R_{n-1}^{(q)}}}{1} + \frac{\dfrac{c_{q+1}}{R_{n-1}^{(q)}R_{n-2}^{(q)}}}{1} + \ldots + \frac{\dfrac{c_{q+1}}{R_1^{(q)}R_0^{(q)}}}{1}. \tag{20}$$

By virtue of the relation $c_{q+1} \in \Omega_{q+1}$, we conclude that, for $n \ge 0$, the elements of the fractions $h_n^{(q+1)}$ satisfy the condition

$$\left|\frac{c_{q+1}}{R_n^{(q)}R_{n-1}^{(q)}}\right| \le \frac{|c_{q+1}|}{r_1\prod_{j=2}^{q}\xi_j^2} = \frac{|c_{q+1}|}{\prod_{j=1}^{q} r_j} < \frac{1}{4}. \tag{21}$$

According to Lemma 2, for

$$|a| = \frac{|c_{q+1}|}{\prod_{j=1}^{q} r_j},$$

we obtain the inequality $\left|h_n^{(q+1)}\right| > \xi_{q+1}$, where $\xi_{q+1} = \frac{1}{2}(1+d_{q+1})$. Hence, the inequalities

$$\left|R_n^{(q+1)}\right| = \left|R_n^{(q)}\right|\left|h_n^{(q+1)}\right| \geq K_q \frac{1+d_{q+1}}{2} = K_{q+1}$$

hold for $n \geq 0$.

To estimate the rate of convergence for fraction (11), we use relation (14). We now establish the upper bounds for the expressions

$$\frac{|c_j|^{k_j}}{\prod_{r=1}^{k_j}\left(\left|R_{s_j+m-r}^{(j)}\right|\left|\hat{R}_{s_j-r}^{(j)}\right|\right)}, \qquad j = 1,\ldots,N, \qquad s_j = n - \sum_{\ell=j+1}^{N} k_\ell . \tag{22}$$

For $j = 1$, in view of relation (5), we get

$$\frac{|c_j|^{k_1}}{\prod_{r=1}^{k_1}\left(\left|R_{s_1+m-r}^{(1)}\right|\left|\hat{R}_{s_1-r}^{(1)}\right|\right)} \leq M_1\left(\frac{|c_1|}{|x_1|^2}\right)^{k_1}.$$

If x_1 and y_1 are solutions of the equation $\omega^2 - \omega - c_1 = 0$, then

$$\frac{|c_1|}{|x_1|^2} = \frac{|x_1||y_1|}{|x_1|^2} = p_1$$

and $p_1 < 1$.

In view of the relations $\left|R_n^{(j)}\right| = \left|R_n^{(j-1)}\right|\left|h_n^{(j)}\right|$, $j = 2,\ldots,N$, and inequalities (21), we find

$$\frac{|c_j|}{\left|R_n^{(j)}\right|\left|R_{n-1}^{(j)}\right|} = \frac{\dfrac{|c_j|}{\left|R_n^{(j-1)}R_{n-1}^{(j-1)}\right|}}{\left|h_n^{(j)}\right|\left|h_{n-1}^{(j)}\right|} < \frac{1/4}{\xi_j^2} < \frac{1}{(1+d_j)^2} = p_j .$$

Since $1 \leq k_j \leq n+1$, expressions (22) satisfy the inequalities

$$\prod_{r=1}^{[k_j/2]} \frac{|c_j|}{\left(\left|R_{s_j+m-2r+1}^{(j)}\right|\left|R_{s_j+m-2r}^{(j)}\right|\right)} \prod_{r=1}^{[k_j/2]} \frac{|c_j|}{\left(\left|\hat{R}_{s_j-2r+1}^{(j)}\right|\left|\hat{R}_{s_j-2r}^{(j)}\right|\right)} \leq M_j p_j^{k_j}, \tag{23}$$

where

$$M_j = \max\left\{1, \frac{|c_j|}{p_j K_j^2}\right\} = \frac{|c_j| 4^j}{p_j \cos^2\alpha \prod_{m=2}^{j}(1+d_m)^2}.$$

Hence,

$$|F_{n+m} - F_n| \le \frac{1}{K_N^2} \sum_{k \in I_{n+1}^{(N)}} \prod_{j=1}^{N} M_j p_j^{k_j} \le \frac{\prod_{j=1}^{N} M_j}{K_N^2} \sum_{k \in I_{n+1}^{(N)}} p^{k_1+k_2+\ldots+k_N} = C_{n+N-1}^{N-1} L p^{n+1}.$$

Passing to the limit as $m \to \infty$, we arrive at estimate (19).

We now find the value of fraction (11). By using Proposition 1 in [1, p. 9], we obtain

$$F = \lim_{n\to\infty} F_n = \lim_{n\to\infty} (R_n^{(N)})^{-1} = \lim_{n\to\infty} (h_n^{(1)})^{-1} \cdot (h_n^{(2)})^{-1} \cdot \ldots \cdot (h_n^{(N)})^{-1}.$$

By induction on q, we now prove that $\lim\limits_{n\to\infty} h_n^{(q)} = x_q$, where

$$x_q = \frac{1}{2}\left(1 + \sqrt{1 + \frac{4c_q}{\prod_{p=1}^{q-1} x_p^2}}\right).$$

For $q = 1$, we have $\lim\limits_{n\to\infty} h_n^{(1)} = x_1$, where x_1 is given by relation (3). By the assumption of induction, we find

$$\lim_{n\to\infty}\left|\frac{c_{q+1}}{R_n^{(q)} R_{n-1}^{(q)}}\right| = \frac{|c_{q+1}|}{\prod_{j=1}^{q} |x_j|^2}.$$

According to Lemma 2, the inequalities $\left|h_n^{(s)}\right| > \xi_s$, $n \ge 0$, are true. This enables us to conclude that $\lim\limits_{n\to\infty}\left|h_n^{(s)}\right| = |x_s| \ge \xi_s$, $s = 2,\ldots,q$. In view of the estimates

$$\frac{|c_{s+1}|}{\prod_{j=1}^{s} |x_j|^2} \le \frac{|c_{s+1}|}{|x_1|^2 \prod_{j=2}^{s} \xi_j^2} \le \frac{|c_{s+1}|}{\prod_{j=1}^{s} r_j} < \frac{1}{4}$$

and Theorem 4.1 in [8, p. 47], we obtain $\lim\limits_{n\to\infty} h_n^{(q+1)} = x_{q+1}$. Hence, the value of fraction (11) is $F = \left(\prod\limits_{j=1}^{N} x_j\right)^{-1}$.

The theorem is proved.

Let

$$\partial G = \left\{ z \in \mathbb{C} : \left|\arg\left(z + \frac{1}{4}\right)\right| = \pi \right\},$$

and let K be an arbitrary compact set $(K \subset G)$, where G is given by relation (4). Let

$$\Delta = \max_{z \in \partial K} \left| z + \frac{1}{4} \right|, \quad \delta = \min_{z \in \partial K} \left| z + \frac{1}{4} \right| = \operatorname{dist}\left(-\frac{1}{4}, \partial K \right), \quad \operatorname{dist}(\partial G, z^*) = \operatorname{dist}(\partial G, \partial K),$$

and

$$\varepsilon = \begin{cases} \pi - \arg z^*, & \arg z^* \neq \pi, \\ \pi, & \arg z^* = \pi. \end{cases}$$

We now construct a domain D_1 of the form (6) with the parameters specified above. It is clear that $K \subseteq D_1$.

Theorem 2. *Let the elements of fraction (11) belong to the domains* D_j, *i.e.,* $c_j \in D_j$, $j = 1, \ldots, N$, *where* D_1 *is the domain of the form (6) defined above,*

$$D_j = \left\{ z \in \mathbb{C} : |z| < \frac{m_1^2}{4^{j-1}} \right\}, \quad j = 2, \ldots, N,$$

and

$$m_1 = (1 - \rho_1) \sqrt{\frac{1}{4} + \sqrt{\delta} \sin \frac{\varepsilon}{2} + \delta}, \tag{24}$$

where ρ_1 *is given by relation (8).*

Then:

(i) *fraction (11) converges uniformly in* $D_1 \times D_2 \times \ldots \times D_N$;

(ii) *the following estimate for the rate of convergence is true:*

$$|F_n - F| \leq C_{n+N-1}^{N-1} L \rho^{n+1}, \quad n \geq 1. \tag{25}$$

Here, $\rho = \max\left\{ \rho_1, \frac{1}{3} \right\}$, $L = \dfrac{12^{N-1} M}{m_1^2}$, $M = \dfrac{(1/2 + \sqrt{\Delta})(1 + \rho_1)}{(1 - \rho_1)}$, *and* F *is the value of fraction (11).*

Proof. By analogy with Theorem 1, we prove by induction that $\left| R_n^{(q)} \right| \geq m_q$, $m_q = \dfrac{m_1}{2^{q-1}}$, $n \geq 0$, $q = 1, \ldots, N$. Since $R_n^{(1)} = f_n$ is the n th approximant of the 1-periodic fraction (2), the estimate $\left| R_n^{(1)} \right| \geq |x_1|(1 - \rho_1)$, $n \geq 0$, is true. Here, x_1 and ρ_1 are given by relations (3) and (8). In view of the inclusion $c_1 \in D_1$, we conclude that $\left| R_n^{(1)} \right| \geq m_1$, where m_1 is given by (24).

Assume that the inequalities $\left|R_n^{(s)}\right| \ge m_s$, $s = 2,\ldots,q$, $n \ge 0$, hold. Since the inequalities

$$\left|\frac{c_{q+1}}{R_n^{(q)} R_{n-1}^{(q)}}\right| \le \frac{\left|c_{q+1}\right|}{m_q^2} < \frac{1}{4}, \qquad n \ge 0,$$

are true for the elements of reversed fractions $h_n^{(q+1)}$ of the form (20), we arrive at the following estimates:

$$\left|R_n^{(q+1)}\right| = \left|R_n^{(q)}\right|\left|h_n^{(q+1)}\right| \ge \frac{1}{2} m_q \ge m_{q+1}, \qquad n \ge 0.$$

In order to determine the rate of convergence for fraction (11), we use relation (14). We estimate expressions (22). In view of the equality $c_1 = x_1 y_1$, for $j = 1$, we arrive at the estimate

$$\frac{\left|c_1\right|^{k_1}}{\prod\limits_{r=1}^{k_1}\left(\left|R_{s_1+m-r}^{(1)}\right|\left|\hat{R}_{s_1-r}^{(1)}\right|\right)} \le M\left|\frac{y_1}{x_1}\right|^{k_1} \le M\rho_1^{k_1},$$

where ρ_1 is the maximum value of the quantity $\left|\dfrac{y_1}{x_1}\right|$ given by relation (8).

In view of the inequalities $\left|\dfrac{c_j}{R_n^{(j-1)} R_{n-1}^{(j-1)}}\right| < \dfrac{1}{4}$ and $\left|h_{n-1}^{(s)}\right| > \dfrac{1}{2}$, we conclude that

$$\frac{\dfrac{\left|c_j\right|}{\left|R_n^{(j-1)} R_{n-1}^{(j-1)}\right|}}{\left|h_{n-1}^{(j)}\right|} < \frac{1}{2}$$

for any j, $j = 2,\ldots,N$. Thus, we arrive at the estimate

$$\frac{\left|c_j\right|}{\left|R_n^{(j)}\right|\left|R_{n-1}^{(j)}\right|} = \frac{\dfrac{\left|c_j\right|}{\left|R_n^{(j-1)} R_{n-1}^{(j-1)}\right|}}{\left|h_{n-1}^{(j)}\right|} \frac{1}{\left|1 + \dfrac{\dfrac{c_j}{R_n^{(j-1)} R_{n-1}^{(j-1)}}}{h_{n-1}^{(j)}}\right|} \le \frac{1}{3} = \rho_j.$$

By analogy with Theorem 1, estimate (23) for $1 \le k_j \le n+1$ takes the form

$$\prod_{r=1}^{[k_j/2]} \frac{\left|c_j\right|}{\left(\left|R_{s_j+m-2r+1}^{(j)}\right|\left|R_{s_j+m-2r}^{(j)}\right|\right)} \prod_{r=1}^{[k_j/2]} \frac{\left|c_j\right|}{\left(\left|\hat{R}_{s_j-2r+1}^{(j)}\right|\left|\hat{R}_{s_j-2r}^{(j)}\right|\right)} \le M_j \rho_j^{k_j},$$

where

$$M_j = \max\left\{1, \frac{|c_j|}{\rho_j m_1^2}, \frac{|c_j|}{\rho_j}, \frac{|c_j|}{\rho_j m_1}\right\} = 3 .$$

Hence,

$$|F_{n+m} - F_n| \le \frac{1}{m_N^2} \sum_{k \in I_{n+1}^{(N)}} \prod_{j=1}^{N} M_j \rho_j^{k_j} \le C_{n+N-1}^{N-1} L \rho^{n+1} .$$

This yields estimate (25) as $m \to \infty$. The theorem is proved.

REFERENCES

1. D. Bodnar and M. Bubnyak, "On the convergence of 1-periodic branched continued fractions of a special form," *Mat. Visn. NTSh.*, **8**, 5–16 (2011).
2. D. I. Bodnar, *Branched Continued Fractions* [in Russian], Naukova Dumka, Kiev (1986).
3. C. Brezinski, *History of Continued Fractions and Padé Approximants,* Springer, Berlin (1991), Springer Ser. in Computational Mathematics, Vol. **12**.
4. A. Cuyt, V. B. Petersen, B. Verdonk, H. Waadeland, and W. B. Jones, *Handbook of Continued Fractions for Special Functions,* Springer, New York (2008).
5. W. B. Jones and W. J. Thron, *Continued Fractions: Analytic Theory and Applications,* Addison-Wesley, Reading, MA (1980), Encyclopedia of Mathematics and its Applications, Edited by G.-C. Rota, Vol. 11.
6. L. Lorentzen and H. Waadeland, *Continued Fractions,* Vol. 1, *Convergence Theory,* Atlantis Press/World Scientific, Amsterdam–Paris (2008).
7. O. Perron, *Die Lehre von der Kettenbrüchen, Band II,* Analytisch-Funktionen Theoretishe Kettenbrüche, Teubner, Stuttgart (1957).
8. W. J. Thron and H. Waadeland, "Modifications of continued fractions. A survey," in: *Analytic Theory of Continued Fractions,* Springer, Berlin (1981), *Lect. Notes in Math.*, **932**, 38–66.
9. H. S. Wall, *Analytic Theory of Continued Fractions,* Van Nostrand, New York (1948).