ISSN 2075-9827 e-ISSN 2313-0210
Carpathian Math. Publ. 2017, **9** (1), 10–18
doi:10.15330/cmp.9.1.10-18

http://www.journals.pu.if.ua/index.php/cmp
Карпатські матем. публ. 2017, Т.9, №1, С.10–18

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CONVERGENCE CRITERION FOR BRANCHED CONTC–NUED FRACTIONS OF THE SPECIAL FORM WITH POSITIVE ELEMENTS

In this paper the problem of convergence of the important type of a multidimensional generalization of continued fractions, the branched continued fractions with independent variables, is considered. This fractions are an efficient apparatus for the approximation of multivariable functions, which are represented by multiple power series. When variables are fixed these fractions are called the branched continued fractions of the special form. Their structure is much simpler then the structure of general branched continued fractions. It has given a possibility to establish the necessary and sufficient conditions of convergence of branched continued fractions of the special form with the positive elements. The received result is the multidimensional analog of Seidel's criterion for the continued fractions. The condition of convergence of investigated fractions is the divergence of series, whose elements are continued fractions. Therefore, the sufficient condition of the convergence of this fraction which has been formulated by the divergence of series composed of partial denominators of this fraction, is established. Using the established criterion and Stieltjes-Vitali Theorem the parabolic theorems of branched continued fractions of the special form with complex elements convergence, is investigated. The sufficient conditions gave a possibility to make the condition of convergence of the branched continued fractions of the special form, whose elements lie in parabolic domains.

Key words and phrases: branced continued fraction of the special form, convergence.

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INTRODUCTION

The convergence problem for continued fractions with positive elements is solved by Seidel's criterion.

Theorem 1 ([9, 12]). *A continued fraction* $b_0 + \overset{\infty}{\underset{n=1}{D}} \frac{1}{b_n}$ *with positive elements converges if, and only if, the series* $\sum\limits_{n=1}^{\infty} b_n$ *diverges.*

Convergence criteria for the continued fractions which elements lie in angular [8], parabolic [1, 4, 6] domains was obtained by Seidel's criterion and Stieltjes-Vitaly Theorem.

Necessary, sufficient, necessary and sufficient conditions for convergence of the branched continued fractions (BCF) with N-branches are establised [3, 10, 11]. But, the analog of Seidel's criterion in following statement is not obtained:

УДК 517.524
2010 *Mathematics Subject Classification:* 11A55, 11J70, 30B70, 40A15.

BCF $b_0 + \mathop{D}\limits_{k=1}^{\infty} \sum\limits_{i_k=1}^{N} \dfrac{1}{b_{i(k)}}$ with positive elements converges if the series $\sum\limits_{k=1}^{\infty} \min\limits_{i(k)} b_{i(k)}$ are divergent.

Establishing the analog of Seidel's criterion for the BCF resulted into construction of different types of BCF, in particular:

$$b_0 + \mathop{D}\limits_{k=1}^{\infty} \sum_{i_k=1}^{i_{k-1}} \frac{a_{i(k)}}{b_{i(k)}} = b_0 + \sum_{i_1=1}^{N} \cfrac{a_{i(1)}}{b_{i(1)} + \sum\limits_{i_2=1}^{i_1} \cfrac{a_{i(2)}}{b_{i(2)} + \ddots}}, \tag{1}$$

where

$$a_{i(k)}, b_{i(k)} \in \mathbb{C}, i(k) \in \mathcal{I},\ \mathcal{I} = \{i(k) = i_1 i_2 \dots i_k \ : 1 \le i_k \le i_{k-1} \le \dots \le i_0;\ k \ge 1;\ i_0 = N\}.$$

This fraction is called the BCF of the special form. There are different convergence ctiteria for this fraction [1, 2, 5].

In the case $b_{i(k)} = 1$, and $a_{i(k)}$ are replaced by $a_{i(k)} z_{i_k}$, this fraction is called a multidimensional regular C-fraction with independent variables. This fraction is analog of the BCF for multiple power series. The condition of the correspondence between multiple power series and regular multidimensional C-fraction with independent variables is established in [7].

The analog of Seidel's criterion for the fraction (1) when $a_{i(k)} = 1$, $b_{i(k)} > 0$, $i(k) \in \mathcal{I}$, and $N = 2$ can be found in [6, 11]. The aim of the paper is to establish the analog of Seidel's criterion for arbitrary natural N. Also, using this criterion, the technique of value and elements sets [3, 9] and Stieltjes-Vitaly Theorem [3], to obtain the parabolic convergence region for the following BCF

$$\left(b_0 + \mathop{D}\limits_{k=1}^{\infty} \sum_{i_k=1}^{i_{k-1}} \frac{a_{i(k)}}{1} \right)^{-1}, \tag{2}$$

where b_0, $a_{i(k)}$ are complex numbers, $i(k) \in \mathcal{I}$.

1 MAIN RESULTS

In this paper, it'll be proved following lemmas for obtaining an analog of Seidel's criterion for the BCF

$$b_0 + \mathop{D}\limits_{k=1}^{\infty} \sum_{i_k=1}^{i_{k-1}} \frac{1}{b_{i(k)}}. \tag{3}$$

Lemma 1.1. *Let the BCF* (3) *with positive elements converges and ε be an arbitrary real positive number. Then exists a natural m, depended of ε, such that for each BCF with positive elements*

$$\widehat{b}_0 + \mathop{D}\limits_{k=1}^{\infty} \sum_{i_k=1}^{i_{k-1}} \frac{1}{\widehat{b}_{i(k)}}, \tag{4}$$

where $\widehat{b}_{i(k)} = b_{i(k)}$ for all $i(k) \in \mathcal{I}, k < m$, the following estimate holds

$$\left| f_n' - f_k' \right| < \varepsilon$$

for all $n, k \ge m$ and f_k' be a kth approximant of BCF (4).

Proof. If f_k be a kth approximant of BCF (3) and the fraction converges, then for all $\varepsilon > 0$ exists $m \geq 2$: $|f_{m-1} - f_{m-2}| < \varepsilon$.

Since $f_k = f_k^{'}, k = 1, 2, ..., m-1$, using the monotonicity properties of approximants of a BCF with positive elements, we have that for all $\varepsilon > 0$ for all $n, k \in \mathbb{N}, n \geq m, k \geq m$,

$$\left| f_n^{'} - f_k^{'} \right| \leq \left| f_{m-1}^{'} - f_{m-2}^{'} \right| = |f_{m-1} - f_{m-2}| < \varepsilon.$$

□

Lemma 1.2. *Let $\Delta_0, \Delta_{i(k)}$ be absolute errors of b_0 and $b_{i(k)}, i(k) \in \mathcal{I}$, respectively. If $\widehat{b}_0 > 0$, $\widehat{b}_{i(k)} > 0$ are approximants of b_0 and $b_{i(k)}$, respectively, then the absolute value of relative error of f_m, mth approximant of the BCF* (3)*, is less or equal to the value*

$$\max_{0 \leq s \leq \left[\frac{m}{2}\right]} \max_{i(2s+1) \in \mathcal{I}} \left\{ \frac{\Delta_{i(2s)}}{b_{i(2s)}}, \frac{\Delta_{i(2s+1)}}{\widehat{b}_{i(2s+1)}} \right\}, \tag{5}$$

where $\Delta_{i_0} = \Delta_0, \Delta_{i(2k+1)} = 0$, if $m = 2k$.

Proof. Let $\delta_{\alpha}^{*} = \dfrac{\alpha - \widehat{\alpha}}{\widehat{\alpha}}$, $\delta_{\alpha} = \dfrac{\widehat{\alpha} - \alpha}{\alpha}$, where $\widehat{\alpha}$ is approximate value of α. If $a > 0$, $\widehat{a} > 0$, $b > 0$, $\widehat{b} > 0$, then: $|\delta_{a+b}| \leq \max\{|\delta_a|, |\delta_b|\}$, $|\delta_{a+b}^{*}| \leq \max\{|\delta_a^{*}|, |\delta_b^{*}|\}$, $\left|\delta_{\frac{1}{a}}\right| = |\delta_a^{*}|$, $|\delta_a^{*}| = \left|\dfrac{\delta_a}{1+\delta_a}\right|$.

Let $\delta_{i(k)}^{(m)}$ is the relative error of calculation of the BCF $b_{i(k)} + \mathop{\mathrm{D}}\limits_{s=k+1}^{m} \sum\limits_{i_s=1}^{i_{s-1}} \dfrac{1}{b_{i(s)}}$. Then the absolute value of relative error of f_m is less or equal to:

$$\max_{i_1} \left\{ |\delta_0|, \left|\delta_{i_1}^{*(m)}\right| \right\} \leq \max_{i_1, i_2} \left\{ |\delta_0|, \left|\delta_{i(1)}^{*}\right|, \left|\delta_{i(2)}^{*(m)}\right| \right\} \leq \max_{i_1, i_2, i_3} \left\{ |\delta_0|, \left|\delta_{i(1)}^{*}\right|, \left|\delta_{i(2)}\right|, \left|\delta_{i(3)}^{*(m)}\right| \right\} \leq$$

$$\leq ... \leq \max_{0 \leq s \leq \left[\frac{m}{2}\right]} \max_{i(2s+1) \in \mathcal{I}} \left\{ \left|\delta_{i(2s)}\right|, \left| \frac{\delta_{i(2s+1)}}{1 + \delta_{i(2s+1)}} \right| \right\} = \max_{0 \leq s \leq \left[\frac{m}{2}\right]} \max_{i(2s+1) \in \mathcal{I}} \left\{ \frac{\Delta_{i(2s)}}{b_{i(2s)}}, \frac{\Delta_{i(2s+1)}}{\widehat{b}_{i(2s+1)}} \right\}.$$

□

Let $\mathcal{I}^{(m)} = \{ i(n) = i_1 i_2 \dots i_n \; : m \leq i_n \leq i_{n-1} \leq ... \leq i_0; \; n \geq 1; \; i_0 = N \}, m = \overline{2, N}$. Let the continued fractions are determined recurrently as follows

$$b_0^{(m)} = b_0^{(m-1)} + \mathop{\mathrm{D}}_{k=1}^{\infty} \frac{1}{b_{m[k]}^{(m-1)}}, \; b_{i(n)}^{(m)} = b_{i(n)}^{(m-1)} + \mathop{\mathrm{D}}_{k=1}^{\infty} \frac{1}{b_{i(n)m[k]}^{(m-1)}}, \; m = \overline{1, N}, \tag{6}$$

$m[k] = \underbrace{mm...m}_{k}$, $i(n) \in \mathcal{I}^{(m+1)}$, with the initial conditions $b_0^{(0)} = b_0, b_{i(k)}^{(0)} = b_{i(k)}$, $i(k) \in \mathcal{I}$, where $b_{i(k)}$ are partial denominators of BCF (3).

Theorem 2 (The multidimensional analog of Seidel's criterion)**.** *BCF* (3) *with positive partial denominators converges if and only if for each $m, 1 \leq m \leq N$, and each $i(n), i(n) \in \mathcal{I}^{(m+1)}$, the following series diverge*

$$\sum_{k=1}^{\infty} b_{m[k]}^{(m-1)}, \sum_{k=1}^{\infty} b_{i(n)m[k]}^{(m-1)}, \tag{7}$$

that elements are determined by (6)*.*

Proof. Necessity. Let the fraction (3) is convergent, then the following sth tail of this fraction converges:

$$r_{i(s)} = b_{i(s)} + \mathop{\mathrm{D}}_{k=s+1}^{\infty} \sum_{i_k=1}^{i_{k-1}} \frac{1}{b_{i(k)}}, i(s) \in \mathcal{I}.$$

The proof of this fact is analogous to the proof of the Theorem 2.1 [3]. In particular, if $i_s = 1$, then the following continued fractions are convergent

$$r_1 = b_1 + \mathop{\mathrm{D}}_{k=2}^{\infty} \frac{1}{b_{1[k]}},\ r_{i(n)1} = b_{i(n)1} + \mathop{\mathrm{D}}_{k=2}^{\infty} \frac{1}{b_{i(n)1[k]}},\ i(n) \in \mathcal{I}^{(2)}. \tag{8}$$

According to Seidel's criterion, the series $\sum\limits_{k=1}^{\infty} b_{1[k]}, \sum\limits_{k=1}^{\infty} b_{i(n)1[k]}, i(n) \in \mathcal{I}^{(2)}$ diverge. Let $b_0^{(1)} = b_0 + \dfrac{1}{r_1}, b_{i(n)}^{(1)} = b_{i(n)} + \dfrac{1}{r_{i(n)1}}, i(n) \in \mathcal{I}^{(2)}$. Consider the BCF of the special form with $(N-1)$-branches:

$$b_0^{(1)} + \mathop{\mathrm{D}}_{k=1}^{\infty} \sum_{i_k=2}^{i_{k-1}} \frac{1}{b_{i(k)}^{(1)}}. \tag{9}$$

We shall show that the convergence of BCF (9) follows from convergence of the fraction (3). Let f_n be the nth approximant of the BCF (3). The approximants of the BCF (9), $\widetilde{f}_n$, are the figured approximants of the fraction (3).

$$\widetilde{f}_n = b_0 + \mathop{\mathrm{D}}_{k=1}^{n} \sum_{i_k=1}^{i_{k-1}} \frac{1}{\widetilde{b}_{i(k)}}, \widetilde{b}_{i(k)} = \begin{cases} b_{i(k)}, & \text{if } k < n \text{ or } k = n, i_n \neq 1; \\ b_{i(n)} + \mathop{\mathrm{D}}\limits_{p=1}^{\infty} \dfrac{1}{b_{i(n)1[p]}}, & \text{if } k = n, i_n = 1. \end{cases}$$

Applying the method suggested in [3], we can show that the following relation for difference $f_n - \widetilde{f}_n$ is valid:

$$f_n - \widetilde{f}_n = (-1)^n \sum_{i_1=1}^{N} \sum_{i_2=1}^{i_1} \dots \sum_{i_n=1}^{i_{n-1}} \frac{b_{i(n)} - \widetilde{b}_{i(n)}}{\prod\limits_{p=1}^{n} \widetilde{Q}_{i(p)}^{(n)} Q_{i(p)}^{(n)}},$$

where

$$Q_{i(n)}^{(n)} = b_{i(n)},\ Q_{i(s)}^{(n)} = b_{i(s)} + \mathop{\mathrm{D}}_{r=s+1}^{n} \sum_{i_r=1}^{i_{r-1}} \frac{1}{b_{i(r)}},\ \widetilde{Q}_{i(n)}^{(n)} = \widetilde{b}_{i(n)},\ \widetilde{Q}_{i(s)}^{(n)} = \widetilde{b}_{i(s)} + \mathop{\mathrm{D}}_{r=s+1}^{n} \sum_{i_r=1}^{i_{r-1}} \frac{1}{\widetilde{b}_{i(r)}},$$

$n = 1, 2, \dots;\ s = \overline{1, n-1};\ i(n) \in \mathcal{I};\ i(p) \in \mathcal{I}$. Obviously $b_{i(n)} - \widetilde{b}_{i(n)} = 0$, if $i_n \neq 1$, and $b_{i(n)} - \widetilde{b}_{i(n)} \leq 0$, if $i_n = 1$. Thus, $(-1)^{n+1}\left(f_n - \widetilde{f}_n\right) > 0$, that is $f_{2r} < \widetilde{f}_{2r} < \widetilde{f}_{2r+1} < f_{2r+1}$.

That is to say, the convergence of the fraction (9) follows from the convergence of the fractoin (3). Analogically as for BCF (3), we conclude that series $\sum\limits_{k=1}^{\infty} b_{2[k]}^{(1)}, \sum\limits_{k=1}^{\infty} b_{i(n)2[k]}^{(1)}, i(n) \in \mathcal{I}^{(3)}$, diverge, and from the convergence of the fraction (9) follows that the fraction $b_0 + \mathop{\mathrm{D}}\limits_{k=1}^{\infty} \sum\limits_{i_k=3}^{i_{k-1}} \dfrac{1}{b_{i(k)}}$, $i(k) \in \mathcal{I}^{(3)}$ converges.

Using the same arguments by $(N-2)$ times, we conclude that series $\sum\limits_{k=1}^{\infty} b_{m[k]}^{(m-1)}$, $\sum\limits_{k=1}^{\infty} b_{i(n)m[k]}^{(m-1)}$ are divergence for each $m : 1 \leq m \leq N-1, i(n) \in \mathcal{I}^{(m+1)}$, also the continued fraction $b_0^{(N-1)} + \mathop{\mathrm{D}}\limits_{k=1}^{\infty} \frac{1}{b_{i(k)}^{(N-1)}}$, $i(k) \in \mathcal{I}^{(N)}$ is convergent. It's equivalent by Seidel's criterion to the divergence of the series $\sum\limits_{k=1}^{\infty} b_{N[k]}^{(N-1)}$. Thus, series (7) diverge.

Sufficiency. By mathematical induction on N, we prove the fact that from diverdgence of the series (7) follows the convergence of the BCF (3).

$N = 1$, the continued fraction with positive elements $b_0 + \mathop{\mathrm{D}}\limits_{k=1}^{\infty} \frac{1}{b_{1[k]}}$ converges by Seidel's criterion, if the series $\sum\limits_{i=1}^{\infty} b_{1[k]}$ is divergent.

$N = 2$, the BCF with positive elements $b_0 + \mathop{\mathrm{D}}\limits_{k=1}^{\infty} \sum\limits_{i_k=1}^{i_{k-1}} \frac{1}{b_{i(k)}}, i(k) \in \mathcal{I}, i_0 = 2$, converges by the Theorem 2.8 [11] if series $\sum\limits_{k=1}^{\infty} b_{1[k]}$, $\sum\limits_{k=1}^{\infty} b_{i(n)1[k]}$, $\sum\limits_{k=1}^{\infty} b_{1[k]}^{(1)}$ diverge.

We suppose that for all $N, N < p$, from the divergence of series (7) follows the convergence of the BCF (3). Consider the convergence of the BCF (3) in the case $N = p$.

$$b_0 + \mathop{\mathrm{D}}_{k=1}^{\infty} \sum_{i_k=1}^{i_{k-1}} \frac{1}{b_{i(k)}}, i(k) \in \mathcal{I}, i_0 = p. \tag{10}$$

If $\sum\limits_{k=1}^{\infty} b_{1[k]} = \infty$, $\sum\limits_{k=1}^{\infty} b_{i(n)1[k]} = \infty, i(n) \in \mathcal{I}^{(2)}$, then continued fractions

$$b_0 + \mathop{\mathrm{D}}_{k=1}^{\infty} \frac{1}{b_{1[k]}}, \tag{11}$$

$$b_{i(n)} + \mathop{\mathrm{D}}_{k=1}^{\infty} \frac{1}{b_{i(n)1[k]}}, i(n) \in \mathcal{I}^{(2)}, \tag{12}$$

converge to the values $b_0^{(1)}$ and $b_{i(n)}^{(1)}$, respectively. We replace, the continued fractions (11) and (12) by it's values, and obtaine BCF of the special form with $(p-1)$-branches

$$b_0^{(1)} + \mathop{\mathrm{D}}_{k=1}^{\infty} \sum_{i_k=2}^{i_{k-1}} \frac{1}{b_{i(k)}^{(1)}}, i(k) \in \mathcal{I}^{(2)}, i_0 = p. \tag{13}$$

Since, the series (7) diverge, for each m, $2 \leq m \leq N$, the fraction (13) converges by the hypotesis of induction. We shall show that the fraction (10) is convergent. Consider the difference between the nth approximant of BCF (10) and (13).

Let $b_0^{(1,n)}$, $b_{i(n)}^{(1,n)}$ be the nth approximant of continued fractions (11) and (12) respectively. Then the nth approximant of BCF (10) may be written as

$$f_n = b_0^{(1,n)} + \mathop{\mathrm{D}}_{k=1}^{n} \sum_{i_k=2}^{i_{k-1}} \frac{1}{b_{i(k)}^{(1,n-k)}}, \ i(k) \in \mathcal{I}^{(2)}.$$

It's the BCF with $(p-1)$-branches. The nth approximant of BCF (13) may be written as

$$\widehat{f}_n = b_0^{(1)} + \mathop{\mathrm{D}}_{k=1}^{n} \sum_{i_k=2}^{i_{k-1}} \frac{1}{b_{i(k)}^{(1)}},\ i(k) \in \mathcal{I}^{(2)}.$$

According to the Lemma 1, form the convergence of the fraction (13) follows that for all $\varepsilon > 0$ exists $m \in \mathbb{N}$ such that for all $n, k \in \mathbb{N}, n \geq 2m+2$ takes place $\left|\widehat{f}_n - g_n\right| < \varepsilon$, where

$$\begin{aligned} g_n &= b_0^{(1)} \\ &+ \sum_{i_1=2}^{p} \frac{1}{b_{i(1)}^{(1)}} + \sum_{i_2=2}^{i_1} \frac{1}{b_{i(2)}^{(1)}} + \ldots + \sum_{i_{2m+1}=2}^{i_{2m}} \frac{1}{b_{i(2m+1)}^{(1)}} + \sum_{i_{2m+2}=2}^{i_{2m+1}} \frac{1}{b_{i(2m+2)}^{(1,n-2m-2)}} + \ldots + \sum_{i_n=2}^{i_{n-1}} \frac{1}{b_{i(n)}^{(1,0)}}. \end{aligned}$$

Next we estimate the value $\left|f_n - \widehat{f}_n\right|$: $\left|f_n - \widehat{f}_n\right| \leq |f_n - g_n| + \left|g_n - \widehat{f}_n\right|$. Using the Lemma 2, we estimate the first term in the right of inequality:

$$|f_n - g_n| \leq \max_{0 \leq s \leq m} \max_{i(2s+1)} \left\{ \frac{\left|b_{i(2s)}^{(1,n-2s)} - b_{i(2s)}^{(1)}\right|}{b_{i(2s)}^{(1)}}, \frac{\left|b_{i(2s+1)}^{(1,n-2s-1)} - b_{i(2s+1)}^{(1)}\right|}{b_{i(2s+1)}^{(1,n-2s-1)}} \right\} \cdot g_n.$$

Since the continued fractions (11) converge, we may choose n, $n \geq 2m+2$, such that for all $i(2s+1) \in \mathcal{I}^{(2)}$, $\left|b_{i(2s)}^{(1,n-2s)} - b_{i(2s)}^{(1)}\right| < \dfrac{\varepsilon}{2A}$, $\left|b_{i(2s+1)}^{(1,n-2s-1)} - b_{i(2s+1)}^{(1)}\right| < \dfrac{\varepsilon}{2A}$, where $A = b_0 + \sum\limits_{i_1=1}^{p} \dfrac{1}{b_{i_1}}$.

Thus, $\left|f_n - \widehat{f}_n\right| < \varepsilon$. From the convergence of the fraction (13) follows the convergence of the fraction (10). □

Since the elements of series (7) are difficult to calculate by the relation (6), it's conviniently to use the following sufficient condition for convergence.

Theorem 3. *BCF* (3) *is divergent, if for each* m, $1 \leq m \leq N$, *and each,* $i(n)$, $i(n) \in \mathcal{I}^{(m+1)}$, *the following series are divergent*

$$\sum_{k=1}^{\infty} b_{m[k]}, \sum_{k=1}^{\infty} b_{i(n)m[k]}. \tag{14}$$

The divergence of the series (14) is suffisient for the divergence of the series (7). We shall use the Theorem 3, to obtain the parabolic convergence domain for the BCF (2).

Lemma 1.3. *Let* $\left\{V_{i(k)}\right\}$ *be the sequense of half-planes*

$$V_{i(k)} = V_{i_k} = \left\{z \in \mathbb{C} : Re\left(ze^{-i\gamma}\right) > -\frac{1}{2i_{k-1}} cos\gamma\right\}, k = 1, 2, 3, \ldots,\ 1 \leq i_k \leq i_{k-1},\ i_0 = N,$$

and

$$E_{i(k)} = E_{i_k} = \left\{z \in \mathbb{C} : |z| - Re\left(ze^{-2i\gamma}\right) < \frac{1}{2i_{k-1}} cos^2\gamma\right\},$$

where $-\dfrac{\pi}{2} < \gamma < \dfrac{\pi}{2}$.

Then $\left\{V_{i(k)}\right\}$ *and* $\left\{E_{i(k)}\right\}$ *are the sequenses of value sets and element sets of the BCF* (2).

The proof of this Lemma is analogous to the proof of the corresponding Theorem 1.5 [3] for the BCF with N-branches.

Theorem 4. *Let the elements of the BCF* (2) *lie in the parabolic domains* $a_{i(k)} \in \mathcal{P}_{i(k)}$, $i(k) \in \mathcal{I}$, *where*

$$\mathcal{P}_{i(k)}(\varepsilon) = \mathcal{P}_{i_k}(\varepsilon) = \left\{ z \in \mathbb{C} : |z| - Rez < \frac{1-\varepsilon}{2i_{k-1}} \right\}, \tag{15}$$

ε *be an arbitrary small real number,* $0 < \varepsilon < 1$.

Then

1) there exist a finite limits of even and odd approximants of the BCF (2);

2) BCF (2) *converges if* $\sum\limits_{k=1}^{\infty} b_{m[k]} = \infty$, $\sum\limits_{k=1}^{\infty} b_{i(n)m[k]} = \infty$ *for each* m, $1 \leq m \leq N$, *and each,* $i(n)$, $i(n) \in \mathcal{I}^{(m+1)}$, *where* $b_{i(k)}$ *is definitely determined by the relations* $\left|a_{i(k)}\right| = \left(b_{i(k)}b_{i(k-1)}\right)^{-1}$, $i(k) \in \mathcal{I}$, $b_{i(0)} = b_0 = 1$;

3) the value region of this fraction is the following circle

$$\mathcal{K} = \{z \in \mathbb{C} : |z-1| \leq 1\}.$$

Proof. Let $a_{i(k)} = \left|a_{i(k)}\right| e^{i\alpha_{i(k)}}$, where $\alpha_{i(k)}$ be an argument of number $a_{i(k)}$, $-\pi < \alpha_{i(k)} \leq \pi$, if $a_{i(k)} \neq 0$.

We determine the function

$$a_{i(k)}(z) = \begin{cases} 0, & \text{if } a_{i(k)} = 0, \\ \left|a_{i(k)}\right| e^{iz\alpha_{i(k)}}, & \text{if } a_{i(k)} \neq 0 \end{cases}$$

in domain $\Omega_\delta = \{z \in \mathbb{C} : |Imz| < \delta, |Rez| < 1+\delta\}$, where δ is an arbitrary real number, such that $(1+\delta)^2 e^{\pi\delta} < (1-\varepsilon)^{-1}$.

We shall show that $a_{i(k)}(z) \in \mathcal{P}_{i(k)}(0)$, $i(k) \in \mathcal{I}$, if $z \in \Omega_\delta$.

If $\alpha_{i(k)} = 0$, then $a_{i(k)}(z) \in \mathcal{P}_{i(k)}(0)$. Let $\alpha_{i(k)} \neq 0$ and $z = x + iy$. From $a_{i(k)} \in \mathcal{P}_{i(k)}(\varepsilon)$, we obtain

$$\left|a_{i(k)}(z)\right| - Rea_{i(k)}(z) < \frac{1-\varepsilon}{2i_{k-1}} e^{\pi\delta} \frac{1-\cos\alpha_{i(k)}x}{1-\cos\alpha_{i(k)}}. \tag{16}$$

If we determine the extrema for the function $\mathcal{M}\left(\alpha_{i(k)}, x\right) = \dfrac{1-\cos\alpha_{i(k)}x}{1-\cos\alpha_{i(k)}}$, where $-\pi < \alpha_{i(k)} \leq \pi$, $\alpha_{i(k)} \neq 0$, $|x| \leq 1+\delta$, we obtain $\sup\left(\mathcal{M}\left(\alpha_{i(k)}, x\right)\right) = (1+\delta)^2$.

Thus, $\left|a_{i(k)}(z)\right| - Rea_{i(k)}(z) < \dfrac{1}{2i_{k-1}}$, that is $a_{i(k)}(z) \in \mathcal{P}_{i(k)}(0)$, $i(k) \in \mathcal{I}$.

Consider the functional BCF

$$\left(1 + \mathop{\mathrm{D}}_{k=1}^{\infty} \sum_{i_k=1}^{i_{k-1}} \frac{a_{i(k)}(z)}{1}\right)^{-1}, \; i(k) \in \mathcal{I}. \tag{17}$$

According to the Lemma 3, where $\gamma = 0$, we obtain that the value set of the reciprocal of the fraction (17) is the half-plane $Rez > \dfrac{1}{2}$. Therefore, all approximants of the BCF (17) depend on the domain $\mathcal{K} = \{z \in \mathbb{C} : |z-1| \leq 1\}$.

Thus, any nth approximant of the (17), $f_n(z)$, is the holomorphic function in domain Ω_δ. We use the Theorem 2.13 (Stieltjes-Vitali Therem [3]) for sequence $\{f_n(z)\}$, where in particular $a = -1$, $b = -2$, and $\Delta = \{z \in \mathbb{C} : Rez = 0, |Imz| < \delta\}$.

If $z \in \Delta$, then we write the BCF (17) in the form

$$\left(1 + \mathop{\mathrm{D}}_{k=1}^{n} \sum_{i_k=1}^{i_{k-1}} \frac{\widetilde{a}_{i(k)}}{1}\right)^{-1}, \; i(k) \in \mathcal{I}, \tag{18}$$

where

$$\widetilde{a}_{i(k)} = \begin{cases} 0, & \text{if } a_{i(k)} = 0, \\ \left|a_{i(k)}\right| e^{-y\alpha_{i(k)}}, & \text{if } a_{i(k)} \neq 0. \end{cases}$$

By equivalence transformstion, we can write the fraction (18), into the form

$$\left(1 + \mathop{\mathrm{D}}_{k=1}^{\infty} \sum_{i_k=1}^{i_{k-1}} \frac{1}{b_{i(k)} e^{\alpha_{i(k)} y}}\right)^{-1}, \; i(k) \in \mathcal{I}, \tag{19}$$

where $b_{i(k)}$ is determined by relations $\left|a_{i(k)}\right| = \left(b_{i(k-1)} b_{i(k)}\right)^{-1}$, $b_{i_0} = 1$, $i(k) \in \mathcal{I}$.

The divergence of the series $\sum\limits_{k=1}^{\infty} b_{m[k]}$, $\sum\limits_{k=1}^{\infty} b_{i(n)m[k]}$ for each m, $1 \leq m \leq N$, and each $i(n)$, $i(n) \in \mathcal{I}^{(m+1)}$, is equivalent to the divergence of the series $\sum\limits_{k=1}^{\infty} b_{m[k]} e^{\alpha_{m[k]} y}$, $\sum\limits_{k=1}^{\infty} b_{i(n)m[k]} e^{\alpha_{i(n)m[k]} y}$. The convergence of the BCF (19) follows from the Theorem 2. Thus, the fraction (18) converges.

Therefore, according to Stieltjes-Vitali Therem, the BCF (17) converges on every compact subset of Ω_δ. In particular, it converges when $z = 1$. This is equivalent to the convergence of the BCF (2).

Using the monotonicity properties of approximants of a BCF with positive elements, we find that finite limits of even and odd approximants of the BCF (2) always exist. □

Analogous, we can prove the following Theorem.

Theorem 5. *Let the elements of the BCF* (2) *lie in the parabolic domains* $a_{i(k)} \in \mathcal{P}_{i(k)}, i(k) \in \mathcal{I}$, *where*

$$\mathcal{P}_{i(k)}(\gamma) = \mathcal{P}_{i_k}(\gamma) = \left\{z \in \mathbb{C} : |z| - Re\left(ze^{-2i\gamma}\right) < \frac{1-\varepsilon}{2i_{k-1}} cos^2\gamma\right\}, \tag{20}$$

ε is an arbitrary small real number, $0 < \varepsilon < 1$.

Then

1) there exist a finite limits of even and odd approximants of BCF (2)*;*

2) BCF (2) *converges if* $\sum\limits_{k=1}^{\infty} b_{m[k]} = \infty$, $\sum\limits_{k=1}^{\infty} b_{i(n)m[k]} = \infty$ *for each* m, $1 \leq m \leq N$, *and each* $i(n)$, $i(n) \in \mathcal{I}^{(m+1)}$, *where* $b_{i(k)}$ *is definitely determined by the relations* $\left|a_{i(k)}\right| = \left(b_{i(k)} b_{i(k-1)}\right)^{-1}$, $i(k) \in \mathcal{I}$, $b_{i(0)} = b_0 = 1$*;*

3) the value region of this fraction is the following circle

$$\mathcal{K}(\gamma) = \left\{z \in \mathbb{C} : \left|z - \frac{e^{-i\gamma}}{cos\gamma}\right| \leq \frac{1}{cos\gamma}\right\}.$$

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Received 05.03.2017

Revised 13.04.2017

Боднар Д.І., Біланик І.Б. *Критерій збіжності гіллястих ланцюгових дробів спеціального вигляду з додатними елементами* // Карпатські матем. публ. — 2017. — Т.9, №1. — С. 10–18.

Досліджується питання збіжності важливого класу багатовимірних узагальнень неперервних дробів – гіллястих ланцюгових дробів (ГЛД) з нерівнозначними змінними. Ці дроби є ефективними при наближенні функцій, заданих кратними степеневими рядами. При фіксованих значеннях змінних вони отримали назву гіллястих ланцюгових дробів спеціального вигляду. Значно простіша структура порівняно із загальними гіллястими ланцюговими дробами дала можливість встановити необхідну і достатню умову їх збіжності у випадку додатних елементів. Отриманий результат є багатовимірним узагальненням критерію збіжності Зейделя для неперервних дробів. Умовою збіжності досліджуваних ГЛД є розбіжність рядів елементами яких є неперервні дроби. Тому доводиться достатня ефективна ознака збіжності, що формулюється через розбіжність рядів складених з частинних знаменників даного ГЛД. Використовуючи встановлену достатню ознаку збіжності та теорему Стілтьєса-Віталі, досліджено параболічні області збіжності для ГЛД спеціального вигляду з комплексними елементами. Встановлена достатня ознака дала можливість послабити умови збіжності ГЛД, елементи котрих лежать в параболічних областях.

Ключові слова і фрази: гіллясті ланцюгові дроби спеціального вигляду, збіжність.