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Methodical instructions and recommendations

for practical lessons in discipline

«THEORY PROBABILITY AND MATHEMATICAL STATISTICS»

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Methodical instructions and recommendations for practical lessons in discipline «Theory Probability and Mathematical Statistics» include examples of solution the tasks in the discipline «*Theory Probability and Mathematical Statistics*»

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Content module 1. Probability.

Content module 1. Probability theory.

Practical lesson 1. (2 hours)
Basic concepts of probability theory
1. Events and their types.
2. Classical definition of the probability of a random event. Probability features.
3. Elements of the probability theory combinatorics.
4. Relative frequency of an accidental event. Statistical probability.
5. Operations over events (algebra of events). Vieen Diagrams. Geometric probability.

Problem 1.1. Before competitions the toss-up among sportsmen is spent. In an urn is 300 counters enumerated from 1 to 300. To discover probability of that at random extracted counter the first sportsman will have at least one digit: a) 8; b) 2.

- a) Event A consists that at random taken counter will have at least one digit 8. A trial - a counter choice. Number of all consequences of a trial (simple events) n = 300 as any of counters can be extracted. All of them are equally possible. Event A happens each time when the counter will have or one, or two digits 8. To count up number of all such consequences of a trial, it is enough to discover number of counters with at least one digit 8 in one hundred (8 cannot be on a place of hundreds) and outcome to increase by three. In particular, in first hundred 8 can meet on a place of units in ten counters (8, 18..., 88, 98), and on a place of tens also in ten (80, 81..., 88, 89). Nevertheless 88 it is counted in the first series, therefore all trial consequences in which event A happens, for first hundred it is equalled 19, and $m = 3 \cdot 19 = 57$.

  According to (1.1) P () = 57/300 = 0,19.

  b) B - the extracted counter has at least one digit 2.

  It is obvious, that n = 300. In first hundred there are 19 counters with at least one digit 2, in the second - 20 (it is completed by number 200), in the third (201, 202..., 299, 300) - 99. So, m = 19 + 20 + 99 = 138 and P (A) = 138/300 = 0,46.

Remark. The received outcomes specify that event B in 2,42 times more possibly in comparison with event A.

Problem 1.2. A coin throw twice on a horizontal firm surface. To discover probability of that at least time will fall out the arms.

- For simplification of reasons we will assume, that two coins (to initial setting of Problem we will come back in a theme «Repeated independent trials») rush.

  Event *And* - the arms falls out at least once. A trial - throwing of two coins. All consequences of a trial: (a;a), (a;i), (i;a), (i;i) where *г* - the arms, i - the inscription, the first position answers outcome of throwing of the first coin, the second - the second. They are equally possible and образовывают a complete group. So, n = 4, m = 3, P (A) = 3/4.

D'Alembert parsed: the arms will appear or at the first throwing, or at the second, or will not appear at all. All consequences three from which assist event A two that is why the required probability is equaled 2/3.

Let's consider one more task which was one of historically the first for which solution classical definition of probability has been used.

Problem 1.3. To discover probability of all possible values of a sum of points on least upper bounds of two thrown playing cubes.

- It is obvious, that the minimum value of the sum is number 2, and maximum - 12. Therefore it will be necessary to discover probabilities of casual events ($\sum$ = 2), ($\sum$ = 3)..., ($\sum$ = 12), where $\sum$ - a sum of points on least upper bounds of cubes. Analyze, why each of these events is the casual. A trial are throwing of two cubes, consequences is the population of pairs points on least upper bounds. We will count up number of these pairs. «Unit »the first cube can meet with 1, 2..., 6 the second, i.e. it" generates"6 consequences of a trial. The same amount« is generated also "by a twain». Thus it is necessary to mean, that trial consequences (1÷2) and (2÷1) - different (the first digit answers number of points on a least upper bound of the first cube, the second - the second). A similar situation with remaining numbers of the first cube. So, $n = 6 \cdot 6 = 36$. Each of these consequences meets the requirements of probability definition: they equally possible and образовывают a complete group. Really, they pairwise incompatible as appearance of one from consequences at a trial eliminates appearance of any another in the same trial. At a trial one of consequences will without fail take place. Eventually, equal possibility it is reached at the expense of symmetry of cubes and a material homogeneity of which they are made.

To event ($\sum = 2$) favour only one consequence (1÷1). Therefore P ($\sum = 2$) = 1/36. To event ($\sum = 3$) favour already two consequence of a trial: (1÷2), (2÷1). Therefore P ($\sum = 3$) = 2/36. Continuing an evaluation of probabilities of remaining casual events, we will receive such table:

k	2	3	4	5	6	7	8	9	10	11	12
$P(\sum = k)$	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

The table analysis allows to draw output, in particular, that probability of event ($\sum = 7$) six times more probabilities of events ($\sum = 2$) and ($\sum = 12$).

Practical lesson 2. (6 hours)

The probability multiplication theorem.
1. Event of production, conditional probability.
2. Probability multiplication theorems for dependent and independent events.
3. Probability of at least one of the events.
Theorems of adding probabilities. Consequences of Theorems.
1. Sum of events.
2. The probability of the sum of two inconsistent events.
3. Opposite events.
4. A complete group of events.
5. The formula of complete probability.
6.Bayes Formula.

Problem 2.1. In a case there are 5 actions of the first view, 6 - the second and 3 - the third. To discover probability of that three at random taken actions will appear the same view.

- Let's designate: A - three at random taken actions are one view, B_i - three taken actions i-th view (i = 1, 2, 3). Event A happens when happen or event B_1, either B_2, or B_3. Other possibilities for appearance A are not present. Such equalities Therefore take place: $A = B_1 + B_2 + B_3$, $P(A) = P(B_1 + B_2 + B_3)$. Events B_1, B_2, B_3 pairwise incompatible and according to a corollary of the theorem 1 (equality (2.9))

$$P(\mathrm{A}) = P(B_1) + P(B_2) + P(B_3).$$

Behind classical definition of probability

$$P(B_1)=\frac{C_5^3}{C_{14}^3}=\frac{5}{182},\ \ P(B_2)=\frac{C_6^3}{C_{14}^3}=\frac{5}{91},\ \ P(B_3)=\frac{C_3^3}{C_{14}^3}=\frac{1}{364}.$$

Definitively

$$P(A)=\frac{5}{182}+\frac{5}{91}+\frac{1}{364}=\frac{31}{364}.$$

Let's show, how it is possible to discover probabilities of events B_1, B_2 and B_3, using the theorem of multiplication of probabilities. With that end in view we will designate: $B_j^{(i)}$ - *j-th* selected action (j = 1, 2, 3) is the action i-th view (i = 1, 2, 3). Then, in particular, B_1 will take place, if all actions (**both** the first, **and** the second, **and** the third) are actions of the first view i.e. when there are all events $B_1^{(1)}, B_2^{(1)}, B_3^{(1)}$.

So, $B_1 = B_1^{(1)} B_2^{(1)} B_3^{(1)}$, $P(B_1) = P\left(B_1^{(1)} B_2^{(1)} B_3^{(1)}\right)$ and according to equality (2.5) for a case $k = 3$

$$P(B_1) = P\left(B_1^{(1)}\right) P_{B_1^{(1)}}\left(B_2^{(1)}\right) P_{B_1^{(1)} B_2^{(1)}}\left(B_3^{(1)}\right) = \frac{5}{14} \cdot \frac{4}{13} \cdot \frac{3}{12} = \frac{5}{182}.$$

Let's mark, that a choice of equality (2.5) in contrast of equality (2.7) stipulated by association $B_1^{(1)}, B_2^{(1)}, B_3^{(1)}$ (prove!).

Similarly

$$P(B_2) = P\left(B_1^{(2)}\right) P_{B_1^{(2)}}\left(B_2^{(2)}\right) P_{B_1^{(2)} B_2^{(2)}}\left(B_3^{(2)}\right) = \frac{6}{14} \cdot \frac{5}{13} \cdot \frac{4}{12} = \frac{5}{91},$$

$$P(B_3) = P\left(B_1^{(3)}\right) P_{B_1^{(3)}}\left(B_2^{(3)}\right) P_{B_1^{(3)} B_2^{(3)}}\left(B_3^{(3)}\right) = \frac{3}{14} \cdot \frac{2}{13} \cdot \frac{1}{12} = \frac{1}{364}.$$

So, the second method of finding of probabilities of events B_1, B_2 and B_3 yields the same outcomes.

Problem2.2. The firm plans to realise deliveries of two views of products. The probability of failure of deliveries for the first view of products makes 0,05, and for the second 0,08. According to the draught contract at violation of periods of deliveries at least one view of production to the producer penal sanctions which lead to unprofitability of production of both views of products are applied. To discover probability of unprofitability of production of these products.

- Let's designate: C - unprofitability of production of both views of production, A and B - failure of delivery of products of the first and second views accordingly. According to a condition of the task event C happens, if there is **either** event A, **or** an event B, **or** events A and B together, i.e. though one of them. Therefore $C = A + B$, where A and B joint events. According to an addition theorem for joint events

$$P(C) = P(A + B) = P(A) + P(B) - P(AB).$$

On condition $P(A) = 0{,}05$, $P(B) = 0{,}08$ also it is possible to consider, that events A

and B independent, i.e. $P(AB) = P(A)\,P(B)$.

Definitively $P(C) = 0{,}05 + 0{,}08 - 0{,}05 \cdot 0{,}08 = 0{,}126$.

The second method of solution. It is possible to present event C through more simple events A and B also thus:

$$C = A\overline{B} + \overline{A}B + AB.$$

Items-events on the right are pairwise incompatible casual events. Really, suppose, that events $A\overline{B}$ and $\overline{A}B$ are joint, i.e. can take place in a trial. Then the inconsistency turns out: event A, in particular, in one trial both happens, and does not happen. This inconsistency specifies in a supposition inaccuracy. Be convinced in paired incompatibility these events, having used Vienne's diagrams. Having used equalities (2.9) both (2.6) and independence of events A and $\overline{B}$, $\overline{A}$ and B, we will receive:

$$\begin{aligned} P(C) &= P(A\overline{B} + \overline{A}B + AB) = P(A\overline{B}) + P(\overline{A}B) + P(AB) = \\ &= P(A)P(\overline{B}) + P(\overline{A})P(B) + P(A)P(B) = \\ &= 0{,}05 \cdot 0{,}92 + 0{,}95 \cdot 0{,}08 + 0{,}05 \cdot 0{,}08 = 0{,}046 + 0{,}076 + 0{,}004 = 0{,}126, \end{aligned}$$

where according to equality (2.11 *)

$$P(\overline{A}) = 1 - P(A) = 1 - 0{,}05 = 0{,}95, \;\; P(\overline{B}) = 1 - P(B) = 1 - 0{,}08 = 0{,}92.$$

The third method. Opposite to event C there is event $\overline{C}$, which consists that owing to realisation of deliveries of production production of both views of products is cost effective. Event $\overline{C}$ happens when **also** the first view of a product are in time delivered, **and** the second, i.e. there are events both $\overline{A}$, and $\overline{B}$. Therefore $\overline{C} = \overline{A}\overline{B}$, $P(\overline{C}) = P(\overline{A})P(\overline{B}) = 0{,}95 \cdot 0{,}92 = 0{,}874$. But agree (2.11*) $P(C) + P(\overline{C}) = 1$, whence $P(C) = 1 - P(\overline{C}) = 1 - 0{,}874 = 0{,}126$.

Outputs. Advantage of the third method, in particular, consists that he allows solving problem for an arbitrary finite number of views of production. As to the answer the discovered probability should be interpreted thus: in 126 cases from each thousand (12,6 %) it is possible to expect unprofitability of production of both views of production. As such probability big enough, the administration needs to take care of reduction of probabilities of failures of deliveries or (and) reduction of penal sanctions.

Problem 2.3. In a sheaf six different keys from which only one can open the lock. The key is at random selected and attempt to open it the lock becomes. The key which has not approached, is not used any more. To discover probability of that for opening it will be used no more than three keys.

– Let's designate: A_k (k = 1, 2,3) - the lock will be open *by k-th* behind order of selection by a key, B - the lock opens after usage of no more than three keys. Event B will take place, if to the lock approaches **or** the first key (happens A_1), **or** the second (thus the first key has not approached - there is event $\overline{A}_1 \cdot A_2$), **or** the third (the first and second keys have not approached - there is event $\overline{A}_1 \cdot \overline{A}_2 \cdot A_3$). I.e. expression B through more simple events A_1, A_2, A_3 expresses so:

$$B = A_1 + \overline{A}_1 A_2 + \overline{A}_1 \overline{A}_2 A_3.$$

For finding $P(B)$ it is necessary to use equality (2.9) as items are pairwise incompatible events, and then the theorem of multiplication of probabilities for dependent events (having calculated $P_{\overline{A}_1}(A_2)$ and $P_{A_1}(A_2)$, be convinced in because events $\overline{A}_1$ and A_2 are dependent):

$$P(B) = P(A_1 + \overline{A}_1 A_2 + \overline{A}_1 \overline{A}_2 A_3) = P(A_1) + P(\overline{A}_1 A_2) + P(\overline{A}_1 \overline{A}_2 A_3) =$$
$$= P(A_1) + P(\overline{A}_1) P_{\overline{A}_1}(A_2) + P(\overline{A}_1) P_{\overline{A}_1}(\overline{A}_2) P_{\overline{A}_1 \overline{A}_2}(A_3) =$$
$$= \frac{1}{6} + \frac{5}{6} \cdot \frac{1}{5} + \frac{5}{6} \cdot \frac{4}{5} \cdot \frac{1}{4} = 0{,}5.$$

Problem 2.4. In a box there are 11 cards of the cutting alphabet: 4 cards with the character "U", 5 - with character "P" and 2 -"C". Three cards are at random drawn out and decomposed on desktop from left to right. To discover probability of that as a result we will receive a word "cup".

- Let A - obtaining of a word "world", B_1 - the first card has the character "C", B_2 - the second card has the character "U", B_3 - the third - «P». Event A will take place, if takes place **both** event B_1, **and** event B_2, **and** event B_3, i.e. $A = B_1B_2B_3$. Then behind the theorem of multiplication of probabilities for dependent events (equality (2.5) for $k = 3$)

$$P(A) = P(B_1B_2B_3) = P(B_1)P_{B_1}(B_2)P_{B_1B_2}(B_3) = \frac{2}{11}\cdot\frac{4}{10}\cdot\frac{5}{9} = \frac{4}{99}.$$

Remark. This task could be solved and behind classical definition of probability. But trial corollaries (an extraction of three cards) any more are not arrangements, as among 11 initial units is identical (remember arrangement definition!). These corollaries it is impossible to name also and combinations for the same reason (besides, a disposition of characters essential). Therefore for an evaluation m and n it is necessary to use more complicated formulas of combinatorial analysis.

Practical lesson 3. (4 hours)
Repeated independent tests.
1. Scheme Bernoulli.
2. Laplace's local and integral formulas.
3. The Poisson formula.
4. The most likely occurrence of the event.
5. Probability of deviation of relative frequency from probability.

Problem 3.1. The company owns a web of dealers at a stock exchange. Probability of that the dealer will play successfully, makes 0,7. 1) To discover probability of that from five dealers will be in losses: a) two; b) at least two (it is considered, that operations of dealers at a stock exchange are independent). 2) To discover the most probable number of dealers which will play successfully, and also probability of such amount.

– Trial - game of the dealer. As dealers there is 5, n = 5. 1) An event A - unprofitable game of the dealer. On condition $P(\overline{A}) = q = 0{,}7,$ then p = 1 - q = 0,3. For a case a) m

= 2, also it is necessary to discover number of appearance of event $P_5(2)$. As n = 5 - small we use Bernoulli formula:

$$P_5(2) = C_5^2 (0{,}3)^2 \cdot (0{,}7)^3 = 10 \cdot 0{,}09 \cdot 0{,}343 = 0{,}3087.$$

b) $P_5(m \geq 2) = P_5(2 \leq m \leq 5)$ - required probability. Though it visually reminds the left part of the integral Laplace formula, last to use it is impossible, as $npq = 5 \cdot 0{,}3 \cdot 0{,}7$ = = 1,05 <<9. Casual event ($2 \leq m \leq 5$) can take place when or (m = 2), or (m = 3), or (m = 4), or (m = 5), i.e. ($2 \leq m \leq 5$) = (m = 2) + (m = 3) + (m = 4) + (m = 5). Casual events-composed on the right are pairwise incompatible, therefore according to the theorem of the sum of probabilities

$$P_5(2 \leq m \leq 5) = P(m = 2) + P(m = 3) + P(m = 4) + P(m = 5) = P_5(2) + P_5(3) + P_5(4) + P_5(5),$$

Where in the last equality that casual event (m = 2) - in five trials event A will take place exactly two times has been considered, in particular, i.e. $P(m = 2) = P_5(2)$. Having used Bernoulli formula, we will receive:

$$P_5(3) = C_5^3 (0{,}3)^3 \cdot (0{,}7)^2 = 0{,}1323;$$
$$P_5(4) = C_5^4 (0{,}3)^4 \cdot 0{,}7 = 0{,}02835;$$
$$P_5(5) = C_5^5 (0{,}3)^5 \cdot 0{,}7^0 = 0{,}00243.$$

Definitively

$$P_5(2 \leq m \leq 5) = 0{,}3087 + 0{,}1323 + 0{,}028535 + 0{,}00243 = 0{,}47178.$$

The second method. Events ($m \geq 2$) and (m <2) opposite, therefore

$$P_5(m \geq 2) = 1 - P_5(m < 2) = 1 - [P_5(0) + P_5(1)] =$$

$$= 1 - [C_5^0 (0{,}3)^0 (0{,}7)^5 + C_5^1 (0{,}3)^1 (0{,}7)^4] = 1 - (0{,}16807 + 0{,}36015) = 0{,}47178.$$

Output: the second method carries on to the purpose much faster, its efficiency is even more notable at magnification of number of items.

2) Event A - successful game of dealers. On a task condition n = 5, p = 0,7, q = 0,3. The most probable number m_0 dealers who will play successfully, we will discover from a double inequality (3.5)

$$np - q \le m_0 \le np + p.$$

Having substituted value in the left and right parts, we will discover $3{,}2 \le m_0 \le 4{,}2$, whence taking into account that m_0 - an integer, we will definitively receive: $m_0 = 4$. Eventually,

$$P_5(m_0) = P_5(4) = C_5^4 (0{,}7)^4 \cdot 0{,}3 = 0{,}36015.$$

For some tasks the probability of appearance of event A in one trial is rather more complicatedly.

Problem 3.2. Two machine tools with programmed control produce one-type details which arrive on the common pipeline. Their efficiencies concern as 2:3, and the first produces 35 % of details of the first-rate quality with which products for the export are completed, the second - 10 %. To discover probability of that from 400 details of the first-rate quality at random selected from the pipeline it will appear: a) 75 details; b) at least 80; c) it is no more 75.

– Trial - detail selection, on a condition $n = 400$. A - the selected detail has the first-rate quality. We will discover $P\ (A)$ = the Selected detail (the first-rate quality or not) can be made or the first machine tool (event B_1), or the second (event B_2). These hypotheses form a complete group, and event A can take place only after appearance of one from them. Therefore $P\ (A)$ it is possible to discover behind the composite probability formula

$$P(A) = P(B_1)P_{B_1}(A) + P(B_2)P_{B_2}(A).$$

If for some period the first machine tool makes $2a$ details, the second - $3a$ details. For classical probability definition

$$P(B_1) = \frac{2a}{5a} = 0{,}4,\ P(B_2) = \frac{3a}{5a} = 0{,}6,$$

$$P_{B_1}(A) = \frac{35}{100} = 0{,}35,\ P_{B_2}(A) = \frac{10}{100} = 0{,}1.$$

Having substituted the discovered probabilities in the composite probability formula, we will receive

$P(A) = 0{,}4 \cdot 0{,}35 + 0{,}6 \cdot 0{,}1 = 0{,}2$. So, $p = 0{,}2$, $q = 1 - p = 0{,}8$.

a) It is necessary to discover P_{400} (75). As $n = 400$ - big, p and q considerable the inequality $npq = 400 \cdot 0{,}2 \cdot 0{,}8 = 64 > 9$ it is necessary to select the local Laplace formula which in this case will give high accuracy of approach also is fulfilled. According to (3.6)

$$x = \frac{m - np}{\sqrt{npq}} = \frac{75 - 400 \cdot 0{,}2}{\sqrt{400 \cdot 0{,}2 \cdot 0{,}8}} = -0{,}63,$$

φ (-0,63) = φ (0,63) = 0,3271 (for tab. 1 of applications),

$$P_{400}(75) \approx \frac{1}{\sqrt{400 \cdot 0{,}2 \cdot 0{,}8}} \varphi(-0{,}63) = \frac{0{,}3271}{8} = 0{,}0410.$$

b) For finding of probability P_{400} ($m \geq 80$) = P_{400} ($80 \leq m \leq 400$) (the remark to algorithm see) it is used the integral Laplace formula, as $npq = 64 > 9$. In correspondence from (3.8)

$$x_1 = \frac{m_1 - np}{\sqrt{npq}} = \frac{80 - 400 \cdot 0{,}2}{\sqrt{400 \cdot 0{,}2 \cdot 0{,}8}} = 0; \; x_2 = \frac{m_2 - np}{\sqrt{npq}} = \frac{400 - 400 \cdot 0{,}2}{\sqrt{400 \cdot 0{,}2 \cdot 0{,}8}} = 40,$$

$$P_{400}(80 \leq m \leq 400) \approx \Phi(40) - \Phi(0) = 0{,}5 - 0 = 0{,}5.$$

At finding Φ(40) was considered that Φ (x) = 0,5 for 5, and Φ (0) was for tab. 3 of applications.

c) Probability P_{400} ($m \leq 75$) = P_{400} ($0 \leq m \leq 75$) it is again calculated behind the integral Laplace formula:

$$x_1 = \frac{0 - 400 \cdot 0{,}2}{8} = -10; \; x_2 = \frac{75 - 400 \cdot 0{,}2}{8} = -0{,}63,$$

$$P_{400}(0 \leq m \leq 75) \approx \Phi(-0{,}63) - \Phi(-10) = \Phi(10) - \Phi(0{,}63) = 0{,}5 - 0{,}2357 = 0{,}2643.$$

In the last equalities the used oddness Φ (x).

Remark. At finding of probabilities in a) also b) it was possible to reach the higher accuracy as exact values x and x_2 are equal-0,625, and φ (0,62) = 0,3292, Φ (0,62) = 0,2324. For this purpose it is necessary to make a linear interpolation:

φ (0,625) = (φ (0,62) + φ (0,625))/2 = (0,3292 + 0,0,3271)/2 = 0,32815,

Φ (0,625) = (Φ (0,62) + Φ (0,63)/2 = (0,2324 + 0,2357)/2 = 0,23405.

Nevertheless the expediency of such approach is defined by requirements to accuracy of the answer, put by a specific target.

It is how much essential to consider **all** conditions of usage of this or that formula, illustrates a solution of the following task.

Problem 3.3. In the private insurance company 15 000 citizens approximately one age and one social group have been insured. The probability of death insured throughout a year on the average makes 0,004. Each insured brings for January, 1st 64 грн. Insurance, and in case of death its relatives receive from the insurance company 10 000 грн. To discover probability of that the company following the results of a year: 1) will suffer damage; 2) will receive profit which is not less 60 000 грн.

– Trial - insurance of the citizen. A - death insured throughout a year. On a task condition n = 15 000, $P(A) = p$ = 0,004.

1) From insurance of citizens the insurance company has received 15 000 · 64 = 960 000 грн. Also will suffer damage, if not most less this sum is paid families of victims throughout a year (even if **only** this sum will be paid company activity will be unprofitable on this view of insurance taking into account current expenditures throughout a year). This casual (complicated) event will take place, if $m \geq 96$, where m - number of appearance of event A, 96 = 960 000: 10 000. I.e., it is necessary to discover $P_{15\ 000}(m \geq 96) = P_{15\ 000}(96 \leq m \leq 15\ 000)$. Event A - improbable, as $p = 0{,}004 \leq 0{,}1$. Nevertheless to use the formula of Poisson (after application addition theorem of probabilities) it is impossible, as $\lambda = np$ = 15 000 · 0,004 = 60 (much more 9). At the same time, npq = 59,76> 25. Therefore it is possible to use the integral Laplace formula which will give very high accuracy. We discover:

$$x_1 = \frac{m_1 - np}{\sqrt{npq}} = \frac{96 - 15\,000 \cdot 0{,}004}{\sqrt{59{,}76}} = 4{,}66;\ x_2 = \frac{15\,000 - 60}{\sqrt{59{,}76}} = 1932{,}73,$$

$$P_{15000}(96 \le m \le 15\,000) \approx \Phi(1932{,}73) - \Phi(4{,}66) = 0{,}5 - 0{,}499997 = 0{,}000003.$$

2) We will assume at first, that all main expenditures of the company become covered at the expense of other views of insurance. Then profit which is not less 60 000 грн., the company will receive in a case when in case of death of clients she will spend for payment of the insurance sums no more than 960 000 - 60 000 = 900 000 грн. This casual event happens, if $m \le 90$. Then behind the integral Laplace formula

$$P_{15000}(m \le 90) = P_{15000}(0 \le m \le 90) \approx \Phi(x_2) - \Phi(x_1),$$

where $x_1 = \frac{m_1 - np}{\sqrt{npq}} = \frac{0 - 60}{\sqrt{59{,}76}} = -7{,}76;\ x_2 = \frac{90 - 60}{\sqrt{59{,}76}} = 3{,}88.$

I.e.,

$$P_{15000}(0 \le m \le 90) \approx \Phi(3{,}88) - \Phi(-7{,}76) = 0{,}499944 + 0{,}5 = 0{,}999944.$$

At finding Φ(3,88) the linear interpolation for values x = 3,8 and 4,0 was carried out.

Let now a - the summarised operational expenditure of the company for a year at realisation of the specified view of insurance. Then the profit, is not less 60 000 грн., the company will receive when event $m \le k$, where k - the whole part of number (900 000-)/10 000 will take place. Further behind the integral Laplace formula it is possible to discover required probability $P_{15\,000}(0 \le m \le k)$. It already it will be essential less than the previous probability.

Problem 3.4. At scanning of a text material on the average on each one thousand numerals two erratic. To discover probability of that after scanning of the text in size in 2 500 numerals it will appear erratic: a) six numerals; b) at least six.

– Trial - scanning of a numeral of the text, event A - obtaining of an erratic numeral. On a condition n = 2500, $p = P\,()$ = 0,002.

a) Number of appearance of event m = 6. For finding P_{2500} (6) we will take advantage of the Poisson formula, as n - big, p = 0,002 <<0,1, $\lambda = np = 5 < 9$.

$$P_n(m) \approx \frac{\lambda^m e^{-\lambda}}{m!},\ P_{2500}(6) \approx \frac{5^6 \cdot e^{-5}}{6!} = 0{,}17547.$$

The last value is discovered for function $\frac{\lambda^m e^{-\lambda}}{m!}$ of two variables λ and m in tab. 2 of applications for values λ = 5, m = 6.

b) Casual event ($m \geq 6$) which probability should be discovered, is submitiven through simple events thus:

$$(m \geq 6) = (m = 6) + (m = 7) +... + (m = 2500).$$

To use an addition theorem of probabilities in that case it is practically impossible because in a right member there are 2495 items (!). On the other hand, opposite to event ($m \geq 6$) there is an event (m <6) for which the equality is fulfilled

$$(m < 6) = (m = 0) + (m = 1) + (m = 2) + (m = 3) + (m = 4) + (m = 5).$$

Whence after usage of an addition theorem of probabilities it is received

$$P_{2500}(m < 6) = P_{2500}(0) + P_{2500}(1) + P_{2500}(2) + P_{2500}(3) + P_{2500}(4) + P_{2500}(5).$$

Each of items is calculated under the Poisson formula. In this case these probabilities are for tab. 2 of applications for λ = 5 and m = 0, 1..., 5. I.e.

$$P_{2500}(m < 6) = 0{,}00674 + 0{,}03369 + 0{,}08422 + 0{,}14037 + 0{,}17547 + 0{,}17547 = 0{,}61598.$$

Taking into account contrast of events

$$P_{2500}(m \geq 6) = 1 - P_{2500}(m < 6) = 1 - 0{,}61598 = 0{,}38404.$$

Problem 3.5. The probability of that a kinescope meets premium requirements, is equalled 0,8.

1) For a month of Quality Department of a television factory it is checked up 400 kinescopes. To discover probability of that the relative frequency of manufacture of a kinescope of the premium will deviate its probability modulo no more than 0,09.

12) How many kinescopes it is necessary to check up, that with probability 0,95 it was possible to expect a deviation of a relative frequency of manufacture of a kinescope of the premium from its probability no more than 0,04?

3) For next two months of Quality Department has checked up 900 kinescopes. To discover with probability 0,95 limits in which there will be a number *m* kinescopes of the premium among checked up.

– Trial - kinescope check. Event *A* - a kinescope satisfies to qualities of the premium.

On a task condition *n* = 400, *p* = *P* () = 0,8, *q* = 0,2. Using the formula (3.9)

$$P\left(|m/n-p|\le\varepsilon\right)\approx 2\Phi\left(\varepsilon\sqrt{n/(pq)}\right),$$

Where on a condition ε = 0,09, we will receive

$$P\left(|m/400-0{,}8|\le 0{,}09\right)\approx 2\Phi\left(0{,}09\sqrt{400/0{,}8\cdot 0{,}2}\right)=2\Phi(4{,}5)=2\cdot 0{,}499997=0{,}999994.$$

2) According to a task condition *p* = 0,8, *q* = 0,2, ε = 0,04 and $P\left(|m/n-0{,}8|\le 0{,}04\right)=0{,}95.$ It is necessary to discover *n*.

3 formulas (3.9) we will discover equality

$$2\Phi\left(0{,}04\sqrt{n/(0{,}8\cdot 0{,}2)}\right)=0{,}95,$$

Whence

$$\Phi\left(0{,}1\sqrt{n}\right)=0{,}475.$$

Behind a table of values of a Laplace function $(\Phi(1{,}96)=0{,}475)$ we will discover argument for which the last equality is fulfilled: $0{,}1\sqrt{n}=1{,}96.$ Further $\sqrt{n}=19{,}5,\quad n=384{,}16.$

Considering that *n* - whole, and also behaviour of an error in the approximate equality (3.9), is definitively received: *n* ≥ 385.

3) On a condition *n* = 900, *p* = 0,8, *q* = 0,2, $P\left(|m/n-0{,}8|\le\varepsilon\right)=0{,}95.$ It is necessary to discover boundaries for number *m*. We will discover at first ε, having used according to the formula (3.9) equality

$$2\Phi\left(\varepsilon\sqrt{900/(0,8\cdot 0,2)}\right)=0,95 \text{ or } \Phi(75\varepsilon)=0,475.$$

For tab. 3 of applications we will discover $\Phi(1{,}96)=0{,}475,$ therefore 75 ε = 1,96,

whence ε ≈ 0,03.

Thus, with probability 0,95 deviation of a relative frequency of number of kinescopes of the first-rate quality from probability 0,8 satisfies to an inequality

$$|m/800-0,8|\le 0,03 \text{ or } 0,77\le m/800\le 0,83,$$

And it is definitive $616 \le m \le 664$.

Let's come back to reviewing of problem 1.2, the simplified candidate solution which has been reduced in theme 1.

Problem 3.6. A coin throw twice on a horizontal firm surface. To discover probability of that at least time will fall out the arms.

– Trial - a coin-tossing, event A - arms appearance. It is necessary to discover P_2 ($m \ge 1$). Event ($m \ge 1$) opposite to event ($m = 0$). Therefore P_2 ($m \ge 1$) = 1 - P_2 (0) = $1-C_2^0(1/2)^0(1/2)^2=3/4$. the Received outcome coincides with the answer of problem 1.2.

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Practical lesson 4. (6 hours)

Discrete random variables.

1. Random variables and their types.

2. The law of probability distribution of a discrete random variable.

3. The main distributions of discrete (integer) random variables: binomial, Poisson, uniform, geometric, hypergeometric.

4. Actions on discrete random variables.

5. Mathematical expectation, dispersion, mean square deviation, initial and central moment.

6. Numerical characteristics of binomial distribution.

Problem 4.1. The collector at random takes two details from the container in which there are 20 details among which 4 non-standard. Find the distribution law of number of non-standard details among the selected.

– Let X - number of non-standard details among two selected. Possible values X - 0, 1, 2. We will discover appropriate probabilities, using classical definition:

$$p_1 = P(X=0) = m/n = C_{16}^2 / C_{20}^2 = 12/19,$$

$$p_2 = P(X=1) = C_{16}^1 C_4^1 / C_{20}^2 = 32/95,$$

$$p_3 = P(X=2) = C_4^2 / C_{20}^2 = 3/95.$$

Check: $p_1 + p_2 + p_3 = 12/19 + 32/95 + 3/95 = 1.$

The required distribution law has such appearance:

X	0	1	2
P	12/19	32/95	3/95

For visualisation the distribution law of a discrete random variable occasionally represent and **graphically**. For this purpose in a rectangular Cartesian frame $x0p$ put aside points (x_1, p_1), (x_2, p_2)..., (x_n, p_n), and then each two adjacent points connect rectilinear segments. The received broken line is named **as a probability distribution polygon**.

Problem 4.2. In a sheaf there are five keys from which only one approaches to the lock. To make the distribution law of number of keys which are tested at lock opening if the key which was in a trial, in following trials of involvement does not accept. Find probability of that the number of trials will not exceed two.

– Let's designate: X - number of keys which are tested at lock opening; A_i - casual event which consists that i-th the key will open lock $(i = \overline{1,5})$. Possible values X: 1, 2, 3, 4, 5. We will discover appropriate probabilities of the distribution law, beforehand having become clear structure of casual events (X = 1), (X = 2)..., (X = 5):

$$(X=1)=A_1,\ \ (X=2)=\overline{A}_1A_2,\ (X=3)=\overline{A}_1\overline{A}_2A_3,$$
$$(X=4)=\overline{A}_1\overline{A}_2\overline{A}_3A_4,\ (X=5)=\overline{A}_1\overline{A}_2\overline{A}_3\overline{A}_4A_5.$$

Having used the multiplication theorem for dependent events, we will receive:

$$p_1=1/5,\ \ p_2=P(\overline{A}_1)P_{\overline{A}_1}(A_2)=\frac{4}{5}\cdot\frac{1}{4}=\frac{1}{5},$$
$$p_3=P(\overline{A}_1)P_{\overline{A}_1}(\overline{A}_2)P_{\overline{A}_1\overline{A}_2}(A_3)=\frac{4}{5}\cdot\frac{3}{4}\cdot\frac{1}{3}=\frac{1}{5},$$
$$p_4=\frac{1}{5},\ \ p_5=\frac{1}{5}.$$

Required the distribution looks like

X	1	2	3	4	5
P	1/5	1/5	1/5	1/5	1/5

Casual event ($X \le 2$) consists that the number of trials of keys will not exceed two. Clearly, that ($X \le 2$) = (X = 1) + (X = 2). Owing to an addition theorem of probabilities for incompatible events $P(X \le 2) = P(X = 1) + P(X = 2) = 2/5$.

The equality of all probabilities is prominent feature of the received distribution law. The appropriate random variable is named **evenly distributed**. Generally the integer random variable **is distributed under the uniform law** (**evenly distributed**) if probabilities in the distribution law have such appearance: $p_k=P(X=k)=1/n,\ k=\overline{1,n}.$

3.2. The binomial distribution

Problem 4.3. The probability of appearance of event A in each of n retests is equalled p. To make the distribution law of random variable X as a numbers of appearance of event in n trials.

– Possible values of variable X the such: $x_1 = 0$, $x_2 = 1$, $x_3 = 2$..., $x_{n+1} = n$. Let m any of these numbers. Then ($X = m$) - casual event which consists that in n repeated independent trials (P (A) = p there is invariable for each trial) an event A will happen exactly m time. Then its probability can be designated P_n (m) (labels that 3) and to discover behind Bernoulli formula: $P(X=m)=P_n(m)=C_n^m p^m q^{n-m}$, where m = 0, 1, 2..., n. So, analytical expression of the law of a probability distribution of given random variable X has such appearance:

$$p_{m+1}=P(X=m)=C_n^m p^m q^{n-m}, \qquad (4.3)$$

where m = 0, 1, 2..., n; q = 1 - p.

The distribution law of integral random variable which probabilities are behind Bernoulli formula (4.3), is named **as the binomial**. The motivation of such title is stipulated by that the right member of equality (4.3) can be considered as a common member of expansion of a binominal formula:

$$(p+q)^n = C_n^n p^n + C_n^{n-1} p^{n-1} q + ... + C_n^m p^m q^{n-m} + ... + C_n^0 q^n.$$

Practical lesson 5. (4 hours)
Continuous random variables.
1. Distribution function, distribution density, their interconnection and properties.
2. Mathematical expectation. Dispersion, mean square deviation.

Problem 5.1. Discrete random variable X is set by the law of allocation

X	1	3	6	8
P	0,2	0,4	0,3	0,1

. Find a distribution function and to construct its graph.

– If $x \le 1$, according to property 3 F (x) = 0.

Let $x \in$ (1; 3]. Then casual event (X <x) = (X = 1), as 1 - uniform possible value which is less from x. That is why it agree (5.1) for (1; 3] F (x) = P (X < x) = P (X = 1) = 0,2.

If $x \in$ (3; 6] then event (X < x) happens then, and only when or X = 1, or X = 3, i.e. such equality (X < x) = (X = 1) + (X = 3) where items on the right are incompatible casual events takes place. Usage of an addition theorem of probabilities and definition (5.1) allows to discover *F* (x*)* on this gap:

$$F(x) = P(X < x) = P(X = 1) + P(X = 3) = 0,2 + 0,4 = 0,6.$$

If $x \in (6; 8]$, by analogy to the previous gap

$$F(x) = P(X < x) = P(X = 1) + P(X = 3) + P(X = 6) = 0,9.$$

At last if 8, event $(X < x)$ - authentic, and $F(x) = 1$.

So, the distribution function has such appearance:

$$F(x) = \begin{cases} 0, & \text{якщо } x \le 1, \\ 0,2, & \text{якщо } 1 < x \le 3, \\ 0,6, & \text{якщо } 3 < x \le 6, \\ 0,9, & \text{якщо } 6 < x \le 8, \\ 1, & \text{якщо } x > 8, \end{cases}$$

and its graph is represented on fig. 5.2.

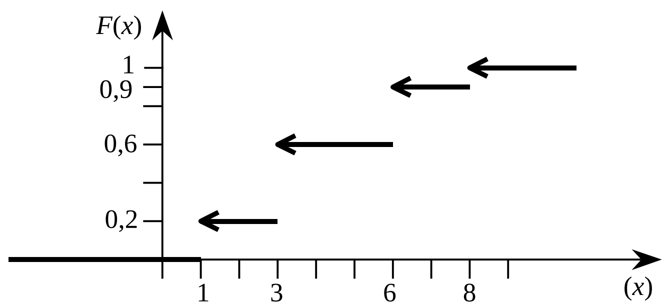

Fig. 5.2.

where arrows mark right-hand ruptures.

Problem 5.2. Random variable X is set by a distribution function

$$F(x) = \begin{cases} 0 & \text{при } x \le -2, \\ x^2/4 + x + 1 & \text{при } -2 < x \le 0, \\ 1 & \text{при } x > 0. \end{cases}$$

Find probability of that: 1) in test data *X* will gather value from an interval (-1; 0); 2) in four trials *X* will gather three times possible value from gap [-4;-1).

– 1) For finding *P* (-1 <*X* <0) we will take advantage of one of formulas (5.4), having supposed $a = 1$, $b = 0$. Then

$P(–1 < X < 0) = F(0) –F(–1) = 0^2/4 + 0 + 1 – ((–1)^2/4 – 1 + 1) = 1 – 1/4 = 0,75.$

2) Beforehand we will discover probability of appearance of event ($-4 \le X <-1$) in one trial, having considered that-4 belongs to the first gap задання to function, in which each point $F(x) = 0$. Again behind the formula (5.4)

$$P\ (-4 \le X <-1) = F\ (-1)-F\ (-4) = 0,25 - 0 = 0,25.$$

Practical lesson 6. (6 hours)

Basic laws of the distribution of continuous random variables.

1. Uniform, normal, index and their numerical characteristics.
2. The probability of falling into an interval and deviation from its mathematical expectation of a normally distributed random variable.

Problem 6.1. Boxes with chocolate are packed automatically, thus the average mass of one box makes 1,04 kg. It is known, that only 2,5 % of boxes have a mass, it is less 1 kg. Assuming, that the mass of boxes is distributed normally, to discover an average standard deviation.

– Let's designate: X - a mass of at random taken box with chocolate. On condition X - normally distributed random variable, $M\ (X) = 1,04$ and $P\ (X <1) = 0,025$. Opposite to event ($X <1$) there is an event ($1 \le X <\infty$), therefore $P\ (1 \le X <\infty) = 1 - P\ (X <1) = 1 - 0,025 = 0,975$. For finding σ we use the formula (6.4), where $a = 1,04$, $\alpha = 1$, $\beta = \infty$, i.e.

$$P(1 \le X < \infty) = 0{,}975 = \Phi\left(\frac{\infty - 1{,}04}{\sigma}\right) - \Phi\left(\frac{1 - 1{,}04}{\sigma}\right),$$

or

$$0{,}975 = 0{,}5 + \Phi\left(\frac{0{,}04}{\sigma}\right),$$

as $\Phi\ (x)$ - odd function and $\Phi\ (x) = 0,5$ for $x > 5$.

Definitively the equation takes such form

$$\Phi\left(\frac{0,04}{\sigma}\right) = 0,475.$$

For tab. 3 of applications Φ (1,96) = 0,475, therefore

$$0,04/\sigma = 1,96, \text{ whence } \sigma = 0,04/1,96 \approx 0,0204.$$

Problem 6.2. On an automatic lathe the bolts, which nominal length produce of 40 mm. In the course of machine tool operation casual deviations from the specified size which are distributed under the normal law with expectation 0 and an average standard deviation of 1 mm are observed. At the control all bolts which sizes differ from nominal more than on tolerance in 2 mm are rejected. Find probability of that at random taken bolt will appear rejected.

– Let random variable *X* - a size deviation of at random taken bolt from nominal. On a condition of task *X* it is distributed under the normal law and *M (X)* = 0, σ *(X)* = 1. It is necessary to find $P(|X| > 2)$. But casual event $(|X| > 2)$ opposite to event $(|X| \le 2)$, therefore $P(|X| > 2) = 1 - P(|X| \le 2)$.

For probability finding in an equality right member it is possible to use the formula (6.5) with such values of parameters: *a* = 0, ε = 2, σ = 1.

Then $P(|X| \le 2) = 2\Phi(2) = 2 \cdot 0,4772 = 0,9544, \quad P(|X| > 2) = 1 - 0,9544 = 0,0456.$

Problem 6.3. Random variable *X* is distributed under the normal law with expectation *and* = 0 and an average standard deviation σ. At what value σ probability of what at a trial magnitude *X* will gain possible value from an interval (1, 2), will be maximum?

– Behind the formula (6.4)

$$P(1 < X < 2) = \Phi\left(\frac{2}{\sigma}\right) - \Phi\left(\frac{1}{\sigma}\right) = \frac{1}{\sqrt{2\pi}}\left(\int_0^{2/\sigma} e^{-x^2/2}dx - \int_0^{1/\sigma} e^{-x^2/2}dx\right).$$

This probability is function of one variable σ and for finding of its maximum value we will discover critical points, продифференцировав both parts and having equated to zero a derivative:

$$\frac{d}{d\sigma}P(1 < X < 2) = \frac{1}{\sqrt{2\pi}}\left(e^{-(2/\sigma)^2/2}(2/\sigma)' - e^{-(1/\sigma)^2/2}(1/\sigma)'\right) =$$
$$= \frac{1}{\sqrt{2\pi}}\left(e^{-2/\sigma^2}(-2/\sigma^2) - e^{-1/(2\sigma^2)}(-1/\sigma^2)\right) = 0.$$

From here $2e^{-2/\sigma^2} = e^{-1/2\sigma^2}$. Having taken the logarithm left and a right member behind the foundation e, after simple conversions we will receive:

$$\sigma = \sqrt{3/(2\ln 2)} \approx \sqrt{3/(2\cdot 0{,}693)} \approx 1{,}471.$$

It is possible to check up, that $\frac{d}{d\sigma}P(1 < X < 2)$ at transiting through the discovered point changes a sign with «+» on «-» that is why σ ≈ 1,471 attaches maximum significance of probability of casual event $(1 < X < 2)$.