On Convergence and Truncation Error Bounds of 1-periodic Branched Continued Fraction of the Special Form

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Abstract

Branched continued fractions with non-equivalent variables are natural generalization of C-fractions in solving of the problems of correspondence to multiple power series. We obtain branched continued fractions of the special form if values of variables are fixed. For 1-periodic branched continued fraction of the special form we established the conditions of convergence and uniform convergence, and the truncation error bounds.

1 Introduction

The object of our investigation is 1-periodic branched continued fraction (BCF) of the special form. The research review concerning 1-periodic continued fraction is given in the monographs [11, 14, 15, 16]. The parabola theorems play the important role in the analytic theory of continued fractions and particularly 1-periodic continued fraction. The analogs of parabola theorems were established for the branched continued fraction of the general form with N branches

$$1 + \mathop{\mathrm{D}}_{k=1}^{\infty} \sum_{i_k=1}^{N} \frac{a_{i(k)}}{1} = 1 + \sum_{i_1=1}^{N} \cfrac{a_{i(1)}}{1 + \sum\limits_{i_2=1}^{N} \cfrac{a_{i(2)}}{1 + \ddots}}, \tag{1}$$

where $a_{i(k)} \in \mathbb{C}$, $i(k) = i_1 i_2 \dots i_k$ – multi index $(1 \le i_k \le N,\ k \ge 1)$, by D.I. Bodnar [5], T.M. Antonova [1] and for two-dimensional continued fractions by Kh. Yo. Kuchmins'ka [12]. For the branched contionued fraction of the special form

$$b_0 + \mathop{\mathrm{D}}_{k=1}^{\infty} \sum_{i_k=1}^{i_{k-1}} \frac{a_{i(k)}}{b_{i(k)}} = b_0 + \sum_{i_1=1}^{i_0} \cfrac{a_{i(1)}}{b_{i(1)} + \sum\limits_{i_2=1}^{i_1} \cfrac{a_{i(2)}}{b_{i(2)} + \ddots}}, \tag{2}$$

where $a_{i(k)} \in \mathbb{C}$, $i(k)$ – multi index, $1 \leq i_k \leq i_{k-1}$, $i_0 = N$ – integer, T.M. Antonova [2] proved the convergence of the fraction (2) if $b_{i(k)} = 1$ and elements $a_{i(k)}$ satisfy the following conditions: $\sum_{i_k=1}^{i_{k-1}} \left(|a_{i(k)}| - \Re a_{i(k)}\right) \leq 2t(1-t)$, $|a_{i(k)}| \leq \rho(1-t)^2$, $t < 1/2$, $\rho < 1$ and established other convergence criteria for fractions (1) and (2).

O.Ye. Baran [4] obtained the analog of the parabola theorem for fraction (2) if partial numerators $a_{i(k)}$ belong to respective parabolic regions and partial denominators $b_{i(k)}$ – the half-planes.

Investigating the parabola convergence regions, R.I. Dmytryshyn [10] specified lemma 4.41 [11, p. 100]

$$\Re\frac{u+iv}{x+iy} \geq -\frac{\sqrt{u^2+v^2}-u}{2x} \geq -\frac{p}{c}, \tag{3}$$

where $x \geq c > 0$, $\sqrt{u^2+v^2} - u \leq 2p$, $0 < p \leq 1$, and proved the convergence of multidimensional generalization g-fraction

$$\frac{s_0}{1} + \sum_{i_1=1}^{N} \frac{g_{i(1)} z_1}{1} + \sum_{i_2=1}^{N} \frac{(1-g_{i(1)}) g_{i(2)} z_2}{1} + \sum_{i_3=1}^{N} \frac{(1-g_{i(2)}) g_{i(3)} z_3}{1} + \cdots \tag{4}$$

where $s_0 > 0$, $0 < g_{i(k)} < 1$, $k = \overline{1,\infty}$, $i_p = \overline{1,N}$, $p = \overline{1,k}$, $z \in \mathbb{C}^N$ if the following condition is valid

$$z \in \bigcup_{\alpha \in (-\pi/2;\pi/2)} \left\{ z = (z_1, \ldots, z_N) \in \mathbb{C}^N : \sum_{k=1}^{N} \left(|z_k| - \Re(z_k e^{-2i\alpha})\right) < 2\cos^2\alpha \right\}.$$

He also established the truncation error bounds of fraction (4) at some additional conditions.

2 Main results

We obtain 1-periodic branched continued fractions of the special form fraction if $a_{i(k)} = c_{i_k}$, $b_{i(k)} = 1$ ($1 \leq i_k \leq i_{k-1}$, $k \geq 1$) in fraction (2), that is BCF next form

$$\left(1 + \mathop{\mathrm{D}}_{k=1}^{\infty} \sum_{i_k=1}^{i_{k-1}} \frac{c_{i_k}}{1}\right)^{-1} = \left(1 + \sum_{i_1=1}^{N} \cfrac{c_{i_1}}{1 + \sum\limits_{i_2=1}^{i_1} \cfrac{c_{i_2}}{1 + \ddots}}\right)^{-1}, \tag{5}$$

where c_j – complex numbers ($j = \overline{1,N}$), $i_0 = N$ – integer. The n-th approximant of 1-periodic BCF (5) is

$$F_n = \left(1 + \mathop{\mathrm{D}}_{k=1}^{n} \sum_{i_k=1}^{i_{k-1}} \frac{c_{i_k}}{1}\right)^{-1} \qquad (n \geq 1; F_0 = 1).$$

We define

$$R_n^{(q)} = 1 + \mathop{\mathrm{D}}_{k=1}^{n} \sum_{j_k=1}^{j_{k-1}} \frac{c_{j_k}}{1} = 1 + \sum_{j_1=1}^{j_0} \cfrac{c_{j_1}}{1 + \sum\limits_{j_2=1}^{j_1} \cfrac{c_{j_2}}{1 + \ddots \sum\limits_{j_{n-1}=1}^{j_{n-2}} \cfrac{c_{j_{n-1}}}{1 + \sum\limits_{j_n=1}^{j_{n-1}} \cfrac{c_{j_n}}{1}}}}$$

as n-th tail q-th order of 1-periodic BCF (5) ($q = \overline{1,N}$; $n \geq 1$; $j_0 = q$; $R_0^{(q)} = 1$; $R_n^{(0)} = 1$). Obviously, that the tails $R_n^{(q)}$ ($n \geq 1$, $q = \overline{2,N}$) satisfy following recurrence expression

$$R_n^{(q)} = R_n^{(1)} + \sum_{s=2}^{q} \frac{c_s}{R_{n-1}^{(s)}}. \tag{6}$$

Theorem 1. *Let elements c_j ($j = \overline{1,N}$) of (5) satisfy the condition*

$$(c_1, c_2, \ldots, c_N) \in G = G_1 \times G_2,$$

where

$$G_1 = \{z \in \mathbb{C} : |\arg z| \leq \pi - \varepsilon\},$$

$$G_2 = \left\{(z_2, \ldots, z_N) \in \mathbb{C}^{N-1} : \bigcup_{\gamma \in I_\varepsilon} \left\{\sum_{s=2}^{N} (|z_s| - \Re(z_s e^{-2i\gamma})) \leq l \sin^2 \varepsilon/2\right\}\right\},$$

$I_\varepsilon = \left[-\frac{\pi-\varepsilon}{2}, \frac{\pi-\varepsilon}{2}\right]$, *$l$ and ε – some parameters such as* $0 < \varepsilon < \pi/2$, $0 < l \leq \frac{1}{8}$.

Then

1) *1-periodic BCF (5) converges uniformly on any compact of the set G;*

2) *the value set is*

$$\bigcup_{\gamma \in I_\varepsilon} \left\{z \in \mathbb{C} : \left|z - \frac{2e^{-i\gamma}}{\cos\gamma}\right| \leq \frac{2}{\cos\gamma}\right\}; \tag{7}$$

3) if beside above $c_1 \in G_1 \bigcap \{z \in \mathbb{C} : |z| \leq R\}$ *and*

$$(c_2, c_3, \ldots, c_N) \in G_2 \bigcap \left\{ (z_2, \ldots, z_N) \in \mathbb{C}^{N-1} : \sum_{j=2}^{N} |z_j| \leq C \right\},$$

where R, C *– some positive constants (*$R > \frac{1}{4}\cos\varepsilon$, $C \leq \frac{(1+\sqrt{1-8l})^2}{16}\sin^2(\varepsilon/2)$*),*

a) and also $l < 1/8$, $C < \frac{(1+\sqrt{1-8l})^2}{16}\sin^2(\varepsilon/2)$*, then holds the truncation error bounds of (5)*

$$|F - F_m| < L_1 \cdot \begin{cases} \dfrac{\rho_1^{m+2} - \rho_2^{m+2}}{\rho_1 - \rho_2}, & \text{if } \rho_1 \neq \rho_2, \\ (m+1)\rho^{m+1}, & \text{if } \rho_1 = \rho_2 = \rho, \end{cases}$$

where F *– the value of fraction (5),* $L_1 = \dfrac{16\sqrt{\Delta}}{\sin^2(\varepsilon/2)(1-\rho_1)}$, $d = \frac{1-\sqrt{1-8l}}{1+\sqrt{1-8l}}$, $\Delta = \frac{1}{4} + R$, $\delta = \frac{1}{4}\sin\varepsilon$,

$$\rho_1 = \begin{cases} \sqrt{\dfrac{1 - 4\sqrt{\delta}\sin\theta/2 + 4\delta}{1 + 4\sqrt{\delta}\sin\theta/2 + 4\delta}}, & \text{if } \sin\varepsilon \leq \dfrac{1}{1+4R}; \\ \sqrt{\dfrac{1 - 4\sqrt{\Delta}\sin\theta/2 + 4\Delta}{1 + 4\sqrt{\Delta}\sin\theta/2 + 4\Delta}}, & \text{if } \sin\varepsilon > \dfrac{1}{1+4R}, \end{cases}$$

$$\theta = \arcsin\frac{R\sin\varepsilon}{\sqrt{\frac{1}{16} + R^2 - \frac{1}{2}R\cos\varepsilon}}, \ \rho_2 = \frac{16C}{(1+\sqrt{1-8l})^2\sin^2(\varepsilon/2)},$$

b) or $l = 1/8$*, then we obtain the following truncation error bounds*

$$|F - F_m| < \begin{cases} L_1\varrho^{m+1}\dfrac{(m+1)(m+2)+1}{2(m+1)} & \text{if } C < \dfrac{\sin^2(\varepsilon/2)}{16}, \\ L_2\dfrac{1}{m+1} & \text{if } C = \dfrac{\sin^2(\varepsilon/2)}{16}, \end{cases}$$

where $\varrho = \max\left\{\rho_1; \dfrac{\sin^2(\varepsilon/2)}{16}\right\}$, $L_2 = \dfrac{64\sqrt{\Delta}(1+\rho_1+\rho_1^2)}{\sin^2(\varepsilon/2)(1-\rho_1)^3}$.

Proof. 1. We use multidimensional analog of Stieltjes-Vitali Theorem [5, theorem 2.17, p. 66] for proving uniform convergence of 1-periodic BCF. We are going to investigate the functional fraction following form

$$\left(1+\underset{k=1}{\overset{\infty}{\mathrm{D}}}\sum_{i_k=1}^{i_{k-1}}\frac{z_{i_k}}{1}\right)^{-1} \tag{8}$$

and it's respective the sequence n-th approximants $\{F_n(z)\}_{n=1}^{\infty}$, where $z = (z_1, z_2, ..., z_N)$. We prove that this sequence is bounded uniformly if $z \in G$. In this order we estimate modules of tail $R_n^{(j)}(z)$ of the functional fraction ($n \geq 0$, $j = \overline{1, N}$). Considering that $z_1 \in G_1$, $\gamma \in I_\varepsilon$ and according to the parabola theorem 3.43 [14, p. 151] we obtain

$$\Re(R_n^{(1)}(z)e^{-i\gamma}) \geq \frac{1}{2}\cos\gamma \geq \frac{1}{2}\sin(\varepsilon/2).$$

We consider 1-periodic continued fraction

$$1+\underset{k=1}{\overset{\infty}{\mathrm{D}}}\frac{-2l}{1} \tag{9}$$

and denote f_n – n-th approximant ($n \geq 1,\ f_0 = 1$) of it. We prove by the mathematical induction by n ($n \geq 1$) for every j ($2 \leq j \leq N$), that

$$\Re\left(R_n^{(j)}(z)e^{-i\gamma}\right) \geq \frac{1}{2}\sin(\varepsilon/2)\cdot f_n. \tag{10}$$

For $n = 1$, using (3), leads to

$$\begin{aligned}\Re(R_1^{(j)}(z)e^{-i\gamma}) &= \Re(R_1^{(1)}(z)e^{-i\gamma}) + \sum_{s=2}^{j}\Re(z_s e^{-i\gamma})\\ &\geq \frac{1}{2}\sin(\varepsilon/2) + \sum_{s=2}^{j}\Re\left(\frac{z_s e^{-2i\gamma}}{e^{-i\gamma}}\right)\\ &\geq \frac{1}{2}\sin(\varepsilon/2) - \sum_{s=2}^{j}\frac{(|z_s| - \Re(z_s e^{-2i\gamma}))}{2\Re e^{-i\gamma}}\\ &= \frac{1}{2}\sin(\varepsilon/2)(1-2l) = \frac{1}{2}\sin(\varepsilon/2)\cdot f_1.\end{aligned}$$

By induction hypothesis for k holds: $\Re(R_k^{(j)}(z)e^{-i\gamma}) \geq \frac{1}{2}\sin(\varepsilon/2) \cdot f_k$ $(2 \leq j \leq N)$. We define

$$q_k = \frac{1}{2}\sin(\varepsilon/2) \cdot f_k. \tag{11}$$

Implementing recurrence expressions (6) and induction, we obtain

$$\Re(R_{k+1}^{(j)}(z)e^{-i\gamma}) = \Re(R_{k+1}^{(1)}(z)e^{-i\gamma}) + \sum_{s=2}^{j} \Re\left(\frac{z_s e^{-i\gamma}}{R_k^{(s)}(z)}\right) \geq$$
$$\frac{1}{2}\sin\frac{\varepsilon}{2} - \sum_{s=2}^{j} \frac{(|z_s| - \Re(z_s e^{-2i\gamma}))}{2\Re(R_k^{(s)}(z)e^{-i\gamma})} \geq \frac{1}{2}\sin\frac{\varepsilon}{2} - \frac{\sum_{s=2}^{j}(|z_s| - \Re(z_s e^{-2i\gamma}))}{2q_k}$$
$$= \frac{1}{2}\sin\frac{\varepsilon}{2}\left(1 + \frac{-\,l \cdot \sin(\varepsilon/2)}{q_k}\right) = \frac{1}{2}\sin\frac{\varepsilon}{2} \cdot f_{k+1} = q_{k+1}.$$

Since $2l \leq 2 \cdot \frac{1}{8} = \frac{1}{4}$, then $\frac{1}{2} < f_n \leq 1$ by Worpitzky's Theorem. That is why the following inequalities are valid: $\left|R_n^{(j)}(z)\right| \geq \Re\left(R_n^{(j)}(z)e^{-i\gamma}\right) > \frac{1}{4}\sin(\varepsilon/2)$ for any $\gamma \in I_\varepsilon$. Since $F_n(z) = \left(R_n^{(N)}(z)\right)^{-1}$ we obtain: $F_n(z) \in \left\{z \in \mathbb{C} : |z| < \frac{4}{\sin\varepsilon/2}\right\}$, that guarantee the sequence of $\{F_n(z)\}_{n=1}^{\infty}$ is bounded uniformly.

We prove the convergence of that sequence on the compact $\mathcal{D} = D_1 \times \ldots \times D_N$ of set G, where $D_1 = \left\{z \in \mathbb{C} : |z| \leq \frac{1}{4N}, |\arg z| \leq \pi - \varepsilon\right\}$ and

$$D_j = \left\{z_j \in \mathbb{C} : |z_j| \leq \frac{l\sin^2\varepsilon/2}{4N}\right\}$$

$(j = \overline{2, N})$. Since $z_j \in D_j$ $(j = \overline{1, N})$, then $\sum_{s=2}^{N}(|z_s| - \Re(z_s e^{-2i\gamma})) \leq \sum_{s=2}^{N} 2 \cdot \frac{l\sin^2\varepsilon/2}{4N} < l\sin^2(\varepsilon/2)$, that is $\mathcal{D} \subset G$. The convergence of approximants $F_n(z)$ on the compact $\mathcal{D}$ leads from the multidimensional analog Worpitzky's Theorem [3, p. 35], implementing $|z_s| \leq \frac{1}{4N}$ $(s = \overline{1, N})$. The uniform convergence of fraction (5) on any compact of set G follows from the multidimensional analog of Stiltijes-Vitali Theorem.

2. We prove, that the value region of (5) is the set (7). We consider 1-periodic continued fraction $1 + \underset{k=1}{\overset{\infty}{\mathrm{D}}} \frac{-2l\sin^2(\varepsilon/2)/\cos^2\gamma}{1}$ and denote h_n it's n-th approximant $(n \geq 0,\ h_0 = 1)$.

We can prove by the mathematical induction by n for any j $(2 \leq j \leq N)$ and any $\gamma \in I_\varepsilon$ that following inequalities are valid

$$\Re(R_n^{(j)} e^{-i\gamma}) \geq \frac{1}{2} \cos\gamma \cdot h_n$$

analogically, as inequalities (11).

The elements of n-th approximat h_n $(n \geq 1)$ satisfy the condition: $\frac{2l \sin^2(\varepsilon/2)}{\cos^2\gamma} \leq \frac{2l\cos^2(\pi-\varepsilon)/2}{\cos^2\gamma} \leq 2 \cdot l \leq \frac{1}{4}$, that is $\inf_{n\in\mathbb{N}} h_n = \frac{1}{2}$ and $\Re\left(R_n^{(j)} e^{-i\gamma}\right) \geq \frac{1}{4}\cos\gamma$ $(\gamma \in I_\varepsilon)$. Considering that $F_n = \left(R_n^{(N)}\right)^{-1}$ and $|R_n^{(N)}| \geq \Re(R_n^{(N)} e^{-i\gamma}) \geq \frac{1}{4}\cos\gamma$, we obtain $F_n \in \left\{z \in \mathbb{C} : \left|z - \frac{2e^{-i\gamma}}{\cos\gamma}\right| \leq \frac{2}{\cos\gamma}\right\}$. Since $\gamma \in I_\varepsilon$, then the value of n-th approximant F_n $(n \geq 1)$ belongs to (7).

3. Using the inequality

$$|F_n - F_m| \leq \frac{1}{g_n \cdot g_m} \left[\sum_{k=0}^{m} \frac{C^k}{\prod_{r=1}^{k}(g_{n-r} \cdot g_{m-r})} \left|R_{n-k}^{(1)} - R_{m-k}^{(1)}\right| + \frac{C^{m+1}}{\prod_{r=1}^{m+1}(g_{n-r} \cdot g_{m-r})}\right], \quad (12)$$

where $n > m > 0$, $\sum_{s=2}^{N} |c_s| \leq C$ and $|R_n^{(j)}| \geq g_n$ $(n \geq 0;\ j = \overline{2, N})$, was proved in [9], we estimate the truncation error bounds of fraction (5).

We use uniform the truncation error bounds for estimating tails $R_n^{(1)}$ of (5) $|R_n^{(1)} - R_m^{(1)}| \leq M_1 \rho_1^{m+1}$ $(n > m \geq 0)$ where $M_1 = \frac{4\sqrt{\Delta}}{1-\rho_1}$ and

$$\rho_1 = \begin{cases} \sqrt{\dfrac{1 - 4\sqrt{\delta}\sin\theta/2 + 4\delta}{1 + 4\sqrt{\delta}\sin\theta/2 + 4\delta}}, & \text{if } \delta \cdot \Delta \leq \dfrac{1}{16}; \\ \sqrt{\dfrac{1 - 4\sqrt{\Delta}\sin\theta/2 + 4\Delta}{1 + 4\sqrt{\Delta}\sin\theta/2 + 4\Delta}}, & \text{if } \delta \cdot \Delta > \dfrac{1}{16}, \end{cases}$$

on the set $E = \left\{z \in \mathbb{C} : |\arg(z + \frac{1}{4})| \leq \pi - \theta, \delta \leq |z + \frac{1}{4}| \leq \Delta\right\}$ that was proved in [9].

The values of parameters δ, θ, Δ, what were given in this theorem, were found by elementary calculation provided by condition: $S \subset E$, where $S = \{z \in \mathbb{C} : |z| \leq R, |\arg z| \leq \pi - \varepsilon\}$. Since $\delta \cdot \Delta = \frac{\sin\varepsilon}{16}(1 + 4R)$, then conditions $\delta \cdot \Delta \leq \frac{1}{16}$ and $\sin\varepsilon \leq \frac{1}{1+4R}$ are equivalent and the value ρ_1 is defined as in this theorem (Figure 1).

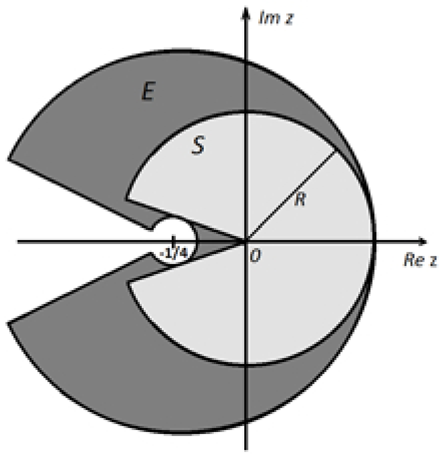

Figure 1: $S \subset E$

3 a. Let $l < \frac{1}{8}$. Using the same scheme as in problem 13 [14, p. 49], we proved, that the value f_n– n-th approximat of 1-periodic continued fraction (9) is equal $f_n = \frac{x^{n+2}-y^{n+2}}{x^{n+1}-y^{n+1}}$ $(n \geq 0)$, where $x = \frac{1+\sqrt{1-8l}}{2}$, $y = \frac{1-\sqrt{1-8l}}{2}$ – the attracting and the repelling fixed points of linear fractional transformation $s(\omega) = 1 - 2l/\omega$. Using inequalities (10) and denotations (11) for $1 \leq k \leq m$ we obtain

$$\frac{C^k}{\prod_{r=1}^{k}(q_{n-r}\cdot q_{m-r})} = \frac{(4C/\sin^2(\varepsilon/2))^k}{\prod_{r=1}^{k}(f_{n-r}\cdot f_{m-r})} = \frac{(4C/\sin^2(\varepsilon/2))^k}{\frac{x^{n+1}-y^{n+1}}{x^{n-k+1}-y^{n-k+1}}\cdot\frac{x^{m+1}-y^{m+1}}{x^{m-k+1}-y^{m-k+1}}}$$

$$= \left(\frac{4C}{x^2\sin^2(\varepsilon/2)}\right)^k \frac{1-(y/x)^{n-k+1}}{1-(y/x)^{n+1}}\frac{1-(y/x)^{m-k+1}}{1-(y/x)^{m+1}}.$$

We denote $f_{-1} = 1$ and for $k = m+1$ the following estimations hold

$$\frac{C^{m+1}}{\prod_{r=1}^{m+1}(q_{n-r}\cdot q_{m-r})} = \frac{(4C/\sin^2(\varepsilon/2))^{m+1}}{\prod_{r=1}^{m+1}(f_{n-r}\cdot f_{m-r})} = \frac{\sin(\varepsilon/2)}{2}\left(\frac{4C}{x^2\sin^2(\varepsilon/2)}\right)^{m+1}$$

$$\frac{1-(y/x)^{n-m}}{1-(y/x)^{n+1}}\cdot\frac{1-(y/x)}{1-(y/x)^{m+1}} < \left(\frac{4C}{x^2\sin^2(\varepsilon/2)}\right)^{m+1}\frac{1-(y/x)^{n-m}}{1-(y/x)^{n+1}}\frac{1-y/x}{1-(y/x)^{m+1}}.$$

Let $C < \frac{(1-8l)\sin^2(\varepsilon/2)}{16}$. We denote $d = y/x$ and, implementing $\frac{1-d^{m-k+1}}{1-d^{m+1}} \leq$

$\frac{1-d^m}{1-d^{m+1}}$ $(1 \le k \le m)$, we obtain

$$\frac{C^k}{\prod_{r=1}^{k}\left(q_{n-k}\cdot q_{m-k}\right)} \le \rho_2^k \frac{1-d^m}{1-d^{m+1}}$$

where $\rho_2 = \dfrac{16C}{(1+\sqrt{1-8l})^2\sin^2(\varepsilon/2)}$. Using the inequality (12), where $g_n = q_n$ $(n \ge 1)$ and $\frac{1-d^m}{1-d^{m+1}} < 1$, let $n \to \infty$ and we obtain the truncation error bounds (5)

$$|F - F_m| \le \frac{16}{\sin^2(\varepsilon/2)}\left(M_1\rho_1^{m+1} + \frac{1-d^m}{1-d^{m+1}}\sum_{k=1}^{m} M_1\rho_1^{m-k+1}\cdot\rho_2^k + \frac{1-d}{1-d^{m+1}}\rho_2^{m+1}\right)$$

$$< L_1 \cdot \begin{cases} \dfrac{\rho_1^{m+2}-\rho_2^{m+2}}{\rho_1-\rho_2}, & \text{if } \rho_1 \ne \rho_2, \\ (m+1)\rho^{m+1}, & \text{if } \rho_1 = \rho_2 = \rho, \end{cases}$$

where $L_1 = \dfrac{16M_1}{\sin^2(\varepsilon/2)} = \dfrac{64\sqrt{\Delta}}{\sin^2(\varepsilon/2)(1-\rho_1)}$.

3 b. Let $l = \frac{1}{8}$. We denote $\widehat{f}_n$ – n-th approximant of 1-periodic continued fraction, which elements are equal $-1/4$. Implementation the formula (3.13) [5], we obtain $\widehat{f}_n = \frac{n+2}{2(n+1)}$ and $\prod_{r=1}^{k} f_{n-r} = \frac{n+1}{2^k(n-k+1)}$. We estimate for $1 \le k \le m$

$$\frac{C^k}{\prod_{r=1}^{k}\left(q_{n-r}\cdot q_{m-r}\right)} = \left(\frac{4C}{\sin^2(\varepsilon/2)}\right)^k \frac{1}{\prod_{r=1}^{k}\left(\widehat{f}_{n-r}\cdot\widehat{f}_{m-r}\right)}$$

$$= \left(\frac{16C}{\sin^2(\varepsilon/2)}\right)^k \frac{(n-k+1)(m-k+1)}{(n+1)(m+1)}$$

and for $k = m+1$

$$\frac{C^{m+1}}{\prod_{r=1}^{m+1}\left(q_{n-k}\cdot q_{m-k}\right)} = \frac{\sin(\varepsilon/2)}{4}\left(\frac{16C}{\sin^2(\varepsilon/2)}\right)^{m+1}$$

$$\frac{n-m}{(n+1)(m+1)} < \left(\frac{16C}{\sin^2(\varepsilon/2)}\right)^{m+1}\frac{n-m}{(n+1)(m+1)}.$$

Let $C < \frac{\sin^2(\varepsilon/2)}{16}$, then let $n \to \infty$ and, implementing $\sum_{k=0}^{m}(m-k+1) = (m+1)(2+m)/2$, we obtain

$$|F - F_m| \le \frac{16M_1}{\sin^2(\varepsilon/2)} \cdot \frac{\sum_{k=0}^{m} \rho_1^{m-k+1}\rho_2^{k}(m-k+1) + \rho_2^{m+1}}{(m+1)}$$
$$< L_1 \varrho^{m+1} \frac{(m+1)(m+2)+1}{2(m+1)},$$

where $\varrho = \max\{\rho_1, \rho_2\}$.

Let $C = \frac{\sin^2(\varepsilon/2)}{16}$, then $\dfrac{C^k}{\prod_{r=1}^{k}(q_{n-r} \cdot q_{m-r})} = \dfrac{(n-k+1)(m-k+1)}{(n+1)(m+1)}$ $(1 \le k \le m)$ and $\dfrac{C^{m+1}}{\prod_{r=1}^{m+1}(q_{n-r} \cdot q_{m-r})} < \dfrac{n-m}{(n+1)(m+1)}$. Let $n \to \infty$ and implement that $\sum_{k=0}^{m} \rho_1^{m-k+1}(m-k+1) + 1 \le \frac{1+\rho_1+\rho_1^2}{(1-\rho_1)^2}$, we obtain

$$|F - F_m| < L_2 \frac{1}{m+1},$$

where $L_2 = \dfrac{16M_1}{\sin^2(\varepsilon/2)} \dfrac{(1+\rho_1+\rho_1^2)}{(1-\rho_1)^2} = \dfrac{64\sqrt{\Delta}(1+\rho_1+\rho_1^2)}{\sin^2(\varepsilon/2)(1-\rho_1)^3}$. □

The truncation error bounds of tails $R_n^{(1)}$ of fraction (5) was established in [9].

$$|R_n^{(1)} - R_m^{(1)}| \le M_1 p_1^{n+1} \quad (n \ge 0), \tag{13}$$

where $M_1 = \frac{4|1+\sqrt{1+4c_1}|}{1-p_1}$ and $p_1 = \left|\frac{1-\sqrt{1+4c_1}}{1+\sqrt{1+4c_1}}\right|$ in the region $\{z \in \mathbb{C} : |\arg(z + 1/4)| < \pi\}$.

Theorem 2. *Let elements c_j $(j = \overline{1, N})$ of fraction (5) satisfy the conditions*

$$c_1 \in G_1 = \{z \in \mathbb{C} : |\arg(z + 1/4)| < \pi\},$$

$$\sum_{s=2}^{N}(|c_s| - \Re(c_s e^{-2i\alpha_1})) \le l\cos^2\alpha_1, \quad l \le \frac{1}{8}, \quad \sum_{s=2}^{N}|c_s| \le C,$$

where

$$2\alpha_1 = \begin{cases} \arg c_1, & \textit{if } \arg c_1 \ne \pi, \\ 0, & \textit{if } \arg c_1 = \pi. \end{cases} \tag{14}$$

Then 1-periodic BCF (5) converges and the truncation error bounds hold

1) if $l < 1/8$ *and* $C < \dfrac{(1+\sqrt{1-8l})^2\cos^2(\alpha_1)}{16}$, *for* $n > m \geq 0$ *we obtain*

$$|F - F_m| < L_1 \cdot \begin{cases} \dfrac{p_1^{m+2} - p_2^{m+2}}{p_1 - p_2}, & \text{if } p_1 \neq p_2, \\ (m+1)p^{m+1}, & \text{if } p_1 = p_2 = p, \end{cases}$$

where $L_1 = \dfrac{32|1+\sqrt{1+4c_1}|}{\cos^2\alpha_1(1-p_1)}$, $p_1 = \left|\dfrac{1-\sqrt{1+4c_1}}{1+\sqrt{1+4c_1}}\right|$, $p_2 = \dfrac{16C}{(1+\sqrt{1-8l})^2\cos^2\alpha_1}$;

2) if $l = 1/8$, *then we obtain the truncation error bounds*

$$|F - F_m| < \begin{cases} L_1 q^{m+1}\dfrac{(m+1)(m+2)+1}{2(m+1)} & \text{if } C < \dfrac{\cos^2\alpha_1}{16}, \\ L_2\dfrac{1}{m+1} & \text{if } C = \dfrac{\cos^2\alpha_1}{16}, \end{cases}$$

where $q = \max\left\{\rho_1; \dfrac{\cos^2\alpha_1}{16}\right\}$, $L_2 = \dfrac{32|1+\sqrt{1+4c_1}|(1+p_1+p_1^2)}{\cos^2\alpha_1(1-p_1)^3}$.

Proof. Analogically, as in the previous theorem we established the following estimates for the tails of (5)

$$\begin{aligned} &\Re(R_n^{(1)}) \geq \tfrac{1}{2}\cos\alpha_1 > 0 \\ &\Re(R_n^{(j)}e^{-i\alpha_1}) \geq \frac{1}{2}\cos\alpha_1 \cdot f_n, \quad (n \geq 1) \end{aligned} \tag{15}$$

where α_1 is defined by formula (14) and f_n – n-th approximant of (9).

Considering the inequality (12) and estimates (15), we obtain the truncation error bounds for (5). □

Conclusions

The uniform convergence and convergence of 1-periodic branched continued fraction of the special form is proved if the element c_1 belongs to some region and sum of the other elemets belogs to union of the parabola-like regions. The truncation error bounds is established at some restrictions of the sum of elements beginning from the second.

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