DOI 10.1007/s10958-018-3828-7

Journal of Mathematical Sciences, Vol. 231, No. 4, June, 2018

DEVELOPMENT OF THE THEORY OF BRANCHED CONTINUED FRACTIONS IN 1996–2016

D. I. Bodnar[1] **and Kh. Yo. Kuchmins'ka**[2]

UDC 517.524

We perform the analysis of investigations in the theory of branched continued fractions carried out for the last 20 years in the directions developed under the general supervision of Prof. V. Ya. Skorobohat'ko (18.07.1927–04.07.1996) including, in particular, the interpolation of functions of several variables by branched continued fractions, the determination of efficient criteria of convergence and computational stability of these fractions, the correspondence between multiple power series and functional branched continued fractions, the investigation of various classes of functional fractions, and the application of branched continued fractions.

In the survey [25] devoted to the bright memory of Prof. V. Ya. Skorobohat'ko (18.07.1927–04.07.1996), the authors analyzed the results obtained in the analytic theory and applications of branched continued fractions (BCF) for the time after the appearance of the first publications [48, 70]. These investigations were carried out under the supervision of Skorobohat'ko in the following directions: the interpolation of functions of several variables by branched continued fractions, the determination of the effective criteria of convergence and computational stability of BCF, the correspondence between multiple power series and functional BCF, the investigation of various classes of functional BCF, the application of BCF to the construction of efficient algorithms for the solution of systems of linear algebraic equations, the construction of the theory of integral continued fractions, and their application to the solution of integral equations. The results of investigations were summarized in the monographs [13, 27, 69, 72].

In the present work, we analyze the development of these directions after 1996. One doctoral-degree thesis [57] and seven candidate-degree theses [10, 28, 32, 33, 36, 42, 61] devoted to the analytic theory of multidimensional continued fractions were defended for this period of time.

The investigations in the field of integral continued fractions were terminated but these fractions were successfully used in the problems of interpolation of nonlinear functionals and operators on continuum sets of the nodes [40, 60].

The problems of interpolation were studied only for special types of BCF, namely, for two- and three-dimensional continued fractions, branched continued fractions with independent variables (Kuchmins'ka [50], Baran [10], and Vozna [59]), and continued fractions (Pahirya [67, 68]).

Nedashkovs'kyi with his colleagues continues the investigations devoted to the construction of numerically stable methods for the solution of systems of linear algebraic equations with the help of BCF [65]. In the monograph [66], the authors solved the systems of linear algebraic equations whose coefficients are polynomials of the parameter.

At the same time, the analytic theory of branched continued fractions with N branches was additionally developed. The most general (at present) criterion of convergence of BCF with positive elements was established

[1] Ternopil National Economical University, Ternopil, Ukraine.

[2] Pidstryhach Institute for Applied Problems in Mechanics and Mathematics, Ukrainian National Academy of Sciences, Lviv, Ukraine.

Translated from Matematychni Metody ta Fizyko-Mekhanichni Polya, Vol. 59, No. 2, pp. 7–18, April–June, 2016. Original article submitted December 2, 2015.

1072-3374/18/2314–0481

in [18]. However, by using this theorem, it is impossible to obtain an analog of the Seidel criterion of convergence of continued fractions [76, 83, 85, 87] at least in the following formulation:

The BCF

$$b_0 + \mathop{\mathrm{D}}_{k=1}^{\infty} \sum_{i_k=1}^{N} \frac{1}{b_{i(k)}}$$

with positive elements converges if the series $\sum_{k=1}^{\infty} \min(b_{i(k)}, i_p = 1,\ldots,N, p = 1,\ldots,k)$ *is divergent.*

Thus, this problem remains open.

In [14, 15], the author analyzed the criteria of convergence of the BCF and formulated unsolved problems. The investigations of convergence of the BCF whose elements belong to coupled and parabolic domains were also additionally developed [1, 2].

The most general parabolic domain of convergence for BCF of the general form was established by Antonova [2].

Theorem 1. *Assume that there exist positive constants* ε, $\varepsilon < 1$, *and* ψ, $\psi < \frac{\pi}{2(1+\varepsilon)}$, *such that, for all possible multiindices of the elements of the BCF*

$$\mathop{\mathrm{D}}_{k=1}^{\infty} \sum_{i_k=1}^{N} \frac{a_{i(k)}}{1}, \tag{1}$$

the following conditions are satisfied

$$\sum_{i_k=1}^{N} \frac{|a_{i(k)}| - \mathrm{Re}(a_{i(k)} \exp(-i(\psi_{i(k-1)} + \psi_{i(k)}))}{\cos\psi_{i(k)} - p_{i(k)}} \le 2(1-\varepsilon) p_{i(k-1)},$$

where $\psi_{i(k)}$ *and* $p_{i(k)}$ *are real numbers such that*

$$|\psi_{i(n)}| \le \psi, \quad n = 0,1,\ldots, \quad 0 \le p_{i(k)} < (1-\varepsilon)\cos\psi_{i(k)}, \quad k = 1,2,\ldots, p_0 \ge 0.$$

Then

(i) the values of all approximants of the BCF (1) are finite and belong to the half plane

$$V_0 = \{w \in C : \mathrm{Re}(w \exp(-i\psi_0)) \ge -p_0\};$$

(ii) there exist finite limits of the subsequences of approximants $\{f_{2n}\}, \{f_{2n-1}\}$ *of the BCF (1);*

(iii) the BCF (1) converges if the series $\sum_{k=1}^{\infty} (\max|a_{i(k)}|)^{-1}$ *diverges.*

Hladun interpreted the problem of stability of BCF via the stability under perturbations as a continuous dependence of infinite BCF on their elements [33, 34]. Let

$$I_0 = \{0\}, \qquad I_k = \{i(k): i_p = 1,2,\ldots,N,\ p = 1,2,\ldots,k\}, \quad k \geq 1.$$

Consider a BCF

$$a_0\left(b_0 + \mathop{\mathrm{D}}_{k=1}^{\infty} \sum_{i_k=1}^{N} \frac{a_{i(k)}}{b_{i(k)}}\right)^{-1} \tag{2}$$

and a perturbed BCF

$$\tilde{a}_0\left(\tilde{b}_0 + \mathop{\mathrm{D}}_{k=1}^{\infty} \sum_{i_k=1}^{N} \frac{\tilde{a}_{i(k)}}{\tilde{b}_{i(k)}}\right)^{-1} \tag{3}$$

with complex elements. A sequence of nonempty sets $\{\Omega_{i(k)}\}$, $\Omega_{i(k)} \subset \mathbb{C}^2$, is called a sequence of sets of absolute stability under perturbations of the BCF (2) if, for any real ε, $\varepsilon > 0$, there exists a real number δ, $\delta > 0$, such that, for any $(a_{i(k)}, b_{i(k)}) \in \Omega_{i(k)}$, $i(k) \in I_k$, $k \geq 0$, and any $(\tilde{a}_{i(k)}, \tilde{b}_{i(k)}) \in \Omega_{i(k)}$, $i(k) \in I_k$, $k \geq 0$, such that $|a_{i(k)} - \tilde{a}_{i(k)}| < \delta$, $|b_{i(k)} - \tilde{b}_{i(k)}| < \delta$, the following inequalities are true:

$$|f_n - \tilde{f}_n| < \varepsilon, \qquad n \geq 0,$$

where f_n and $\tilde{f}_n$ are the nth approximants of the BCF (2) and (3), respectively.

If all $a_{i(k)} \neq 0$, $b_{i(k)} \neq 0$, $i(k) \in I_k$, $k \geq 0$, and, for any $(a_{i(k)}, b_{i(k)}) \in \Omega_{i(k)}$, $i(k) \in I_k$, $k \geq 0$, and any $(\tilde{a}_{i(k)}, \tilde{b}_{i(k)}) \in \Omega_{i(k)}$, $i(k) \in I_k$, $k \geq 0$, such that

$$\left|\frac{a_{i(k)} - \tilde{a}_{i(k)}}{a_{i(k)}}\right| < \delta, \qquad \left|\frac{b_{i(k)} - \tilde{b}_{i(k)}}{b_{i(k)}}\right| < \delta,$$

the inequalities

$$\left|\frac{f_n - \tilde{f}_n}{f_n}\right| < \varepsilon, \qquad n \geq 0,$$

are true, then the sets from the sequence $\{\Omega_{i(k)}\}$ are called the sets of relative stability under perturbations of the BCF (2).

Bodnar and Hladun studied the stability under perturbations of the BCF with positive, real, and in particular, negative or alternating elements, as well as the stability of some subsequences of their approximants [19–21, 34]. The fact of convergence and, especially, of stability in the case where the partial numerators of the BCF are negative turns to be of particular interest [4, 34]. For the continued fractions in the domain $\{x \in R: x < -1/4\}$, these problems have not been investigated yet.

Theorem 2. *Assume that relative errors of elements of the BCF (2) are uniformly bounded. Then the domains*

$$\Omega_0 = (0,+\infty)\times(\nu_0,+\infty), \quad \Omega_{i(k)} = \Omega_k = (0,\mu_k)\times(\nu_k,+\infty), \quad i(k)\in I_k, \quad k\geq 1,$$

where all $\nu_k > 0$ and $\mu_k > 0$, form a sequence of the domains of relative stability of the BCF (2) provided that the series

$$\sum_{k=1}^{\infty} \nu_{k-1}\nu_k^2\mu_k^{-1}(N\mu_{k+1}+\nu_k\nu_{k+1})^{-1}$$

diverges.

The multidimensional sets of stability of the BCF with complex elements were also investigated in the case where

$$(a_{i(k)1}, a_{i(k)2}, \ldots, a_{i(k)N}, b_{i(k)}) \in \Omega_{i(k)}, \quad \Omega_{i(k)} \subset \mathbb{C}^{N+1}, \quad i(k)\in I_k, \quad k\geq 0.$$

The analysis of convergence of the BCF with matrix elements [64] is of interest and quite promising. Let X be the Banach algebra of square matrices of order p over the field $\mathbb{C}$. The matrix BCF is a sequence of approximants

$$F_1 = \sum_{i_1=1}^{N} b_{i_1}^{-1} a_{i_1} = \sum_{i_1=1}^{N} \frac{a_{i_1}}{b_{i_1}},$$

$$F_2 = \sum_{i_1=1}^{N}\left(b_{i_1} + \sum_{i_2=1}^{N} b_{i_1 i_2}^{-1} a_{i_1 i_2}\right)^{-1} a_{i_1} = \mathop{\mathrm{D}}_{k=1}^{2}\sum_{i_k=1}^{N}\frac{a_{i(k)}}{b_{i(k)}}, \ldots,$$

where $a_{i(k)}, b_{i(k)} \in X$ are nondegenerate square $(p\times p)$-matrices.

Theorem 3. *The matrix branched continued fraction*

$$\mathop{\mathrm{D}}_{k=1}^{\infty}\sum_{i_k=1}^{N}\frac{a_{i(k)}}{b_{i(k)}} \tag{4}$$

with elements satisfying the conditions

$$\left\|b_{i(k)}^{-1}\right\| \leq \left(1+\sum_{i_{k+1}=1}^{N}\left\|a_{i(k+1)}\right\|\right)^{-1}, \quad i(k)\in I_k, \quad k\geq 1,$$

is absolutely convergent and

$$\left\{ z \in X : \|z\| \le \sum_{i_1=1}^{N} \left\| a_{i(1)} \right\| \right\}$$

is the set of its values.

For the construction of expansions of the functions of several variables in BCF, it is customary to use two approaches:

(1) determination of recurrence relations for given functions;

(2) the principle of correspondence between the multiple power series and functional BCF.

Actually, the first approach was used by Manzii for the construction of expansions of the ratios of the Appel hypergeometric functions $F_2(a,b,b';c,c';z)$ and $F_3(a,a',b,b';c;z)$. She established new recurrence relations for these functions. In [26, 62, 63], on the basis of these relations, the authors established and studied the correspondence and convergence of expansions of the ratios of these functions in the BCF and estimated the errors of approximations by approximants in certain domains. In the works by Hoyenko [22, 23, 35, 37, 38], the apparatus of branched continued fractions was used for the approximation of the Lauricella hypergeometric functions

$$F_D^{(N)}(a,b_1,b_2,\ldots,b_N;c;z_1,z_2,\ldots,z_N)$$

$$= \sum_{k_1,k_2,\ldots k_N=0}^{N} \frac{(a)_{k_1+k_2+\ldots+k_N}(b_1)_{k_1}(b_2)_{k_2}\ldots(b_N)_{k_N}}{(c)_{k_1+k_2+\ldots+k_N}} \frac{z_1^{k_1} z_2^{k_2} \ldots z_N^{k_N}}{k_1!k_2!\ldots k_N!},$$

where the parameters $a,b_1,b_2,\ldots,b_N$, and c are complex numbers, $c \ne 0,-1,-2,\ldots$; $z_1,z_2,\ldots,z_N$ are complex variables; $(\alpha)_k$ is the Pochhammer symbol.

A multidimensional analog of the Nörlund continued fraction was also constructed and investigated.

Theorem 4. *Assume that the parameters of the function* F_D *are real numbers and satisfy the conditions*

$$a > 0, \quad b_k > 0, \quad k = 1,\ldots,N, \quad 2c > a + \sum_{k=1}^{N} b_k + 1 .$$

The ratio of the Lauricella hypergeometric functions

$$\frac{F_D(a,\bar{b};c;\bar{z})}{F_D(a+1,\bar{b}+\bar{e}_i;c+1;\bar{z})} \tag{5}$$

can be expanded in a Nörlund-type BCF:

$$b_0(\bar{z}) + \mathop{\mathrm{D}}_{k=1}^{\infty} \sum_{i_k=1}^{N} \frac{a_{i(k)}(\bar{z})}{b_{i(k)}(\bar{z})}, \tag{6}$$

where

$$b_0(\overline{z}) = 1 - \frac{a+1}{c} z_1 - \sum_{j=1}^{N} \frac{b_j}{c} z_j, \qquad a_{i(k)}(\overline{z}) = \frac{(a+k)(b_{i_k} + p_{i(k)})}{(c+k-1)(c+k)} z_{i_k} (1 - z_{i_k}),$$

$$b_{i(k)}(\overline{z}) = 1 - \frac{a+k}{c+k} z_{i_k} - \sum_{j=1}^{N} \frac{b_j + p_{i(k)j}}{c+k} z_j, \qquad p_{i(k)} = \sum_{m=1}^{k-1} \delta_{i_k}^{i_m} + \delta_{i_k}^{1}.$$

The BCF (6) uniformly converges on compact sets of the domain

$$G = \left\{ \overline{z} \in \mathbb{C}^N : \operatorname{Re} z_i < \frac{1}{2}, i = 1, \ldots, N \right\}$$

to a holomorphic function obtained as the analytic extension of function (5) holomorphic in a neighborhood of the origin of coordinates onto the domain G.

The approximation of Lauricella–Saran functions by branched continued fractions was studied in [39].

As one of the most efficient and general methods for the expansion of functions both of a single variable [76] and of many variables in functional BCF, we can mention the construction of BCF corresponding to multiple power series.

Let the following formal multiple power series be given

$$L(z) = \sum_{|m(N)| \geq 0} c_{m(N)} z^{m(N)}, \tag{7}$$

where $c_{m(N)} \in \mathbb{C}$, $z = (z_1, z_2, \ldots, z_N) \in \mathbb{C}^N$, $m(N) = m_1 m_2 \ldots m_N$ is a multiindex, $m_i \geq 0$, $i = 1, 2, \ldots, N$, $|m(N)| = m_1 + m_2 + \ldots + m_N$, and $z^{m(N)} = z_1^{m_1} z_2^{m_2} \ldots z_N^{m_N}$.

A functional BCF corresponds to series (7) if the expansion of each its nth approximant in a formal multiple power series coincides with series (7) in all homogeneous polynomials up to the degree ν_n, inclusively, and $\nu_n \to \infty$ as $n \to \infty$.

For the class of multidimensional C-fractions

$$b_0 + \mathop{\mathrm{D}}_{k=1}^{\infty} \sum_{i_k=1}^{N} \frac{a_{i(k)} z_{i_k}}{1},$$

this problem does not have an unambiguous solution. To get the required result, it is necessary to impose additional conditions on the coefficients of the fraction, e.g., the condition that $a_{i(k)}$ do not change under permutations of indices in all multiindices, or to set some $a_{i(k)}$ equal to zero.

Another way is to change the structure of a multidimensional continued fraction. Thus, Kuchmins'ka [49] and Murphy and O'Donohoe [84] defined in this way the first corresponding two-dimensional continued fractions (TDCF) for double power series as follows:

$$\Phi_0 + \mathop{\mathrm{D}}_{i=1}^{\infty} \frac{a_{i,i} xy}{\Phi_i}, \qquad \Phi_i = 1 + \mathop{\mathrm{D}}_{j=1}^{\infty} \frac{a_{i+j,i} x}{1} + \mathop{\mathrm{D}}_{j=1}^{\infty} \frac{a_{i,i+j} y}{1}; \tag{8}$$

moreover, their approximants take the form

$$f_n = \frac{P_n}{Q_n} = \Phi_0^{(n)} + \mathop{\mathrm{D}}_{i=1}^{\left[\frac{n}{2}\right]} \frac{a_{i,i}xy}{\Phi_i^{(n-2i)}(x,y)},$$

$$\Phi_i^{(k)} = 1 + \mathop{\mathrm{D}}_{j=1}^{k} \frac{a_{i+j,i}x}{1} + \mathop{\mathrm{D}}_{j=1}^{k} \frac{a_{i,i+j}y}{1}, \qquad \Phi_i^{(0)} = 1. \tag{9}$$

However, the introduced approximants of TDCF of the form (9) do not allow us to obtain estimates of these fractions with positive elements similar to the estimates for continued fractions, which led to the analysis of the general type of approximants for the TDCF [78, 79]. Thus, for the TDCF

$$\mathop{\mathrm{D}}_{i=0}^{\infty} \frac{a_{i,i}}{\Phi_i}, \qquad \Phi_i = b_{i,i} + \mathop{\mathrm{D}}_{j=1}^{\infty} \frac{a_{i+j,i}}{b_{i+j,i}} + \mathop{\mathrm{D}}_{j=1}^{\infty} \frac{a_{i,i+j}}{b_{i,i+j}}, \tag{10}$$

the general approximants have the form

$$f_n = \mathop{\mathrm{D}}_{i=0}^{n-1} \frac{a_{i,i}}{\Phi_i^{(n-i-1)}},$$

$$\Phi_i^{(k)} = 1 + \mathop{\mathrm{D}}_{j=1}^{k} \frac{a_{i+j,i}}{b_{i+j,i}} + \mathop{\mathrm{D}}_{j=1}^{k} \frac{a_{i,i+j}}{b_{i,i+j}}, \tag{11}$$

$$\Phi_i^{(0)} = b_{i,i}, \qquad n = 1, 2, \ldots.$$

Proposition [78]**.** *Suppose that the elements of the TDCF (10) are positive real numbers and that* j *and* k *are arbitrary natural numbers. Then the approximants* f_n *(11) satisfy the following "fork" property:*

$$f_{2k} < f_{2k+2} < f_{2j+1} < f_{2j-1}.$$

Approximants (9) are figured approximants of the TDCF (8). They are called C-approximants. The TDCF (8) with approximants of type (11) also correspond to a formal double power series [77, 79].

The foundations of the analytic theory of two-dimensional continued fractions were laid in the works by Kuchmins'ka [49–59, 77–82], Antonova and Sus' [5–7], Sus' [59, 71], and Vozna [31, 32, 59, 80].

We now formulate the criterion of convergence of the TDCF established by Sus' in [71].

Theorem 5. *Assume that the elements* b_{ij}, $i, j = 0, 1, \ldots,$ *of the TDCF*

$$\mathop{\mathrm{D}}_{k=0}^{\infty} \frac{1}{b_{kk} + \Phi_k}, \qquad \Phi_k = \mathop{\mathrm{D}}_{j=1}^{\infty} \frac{1}{b_{k+j,k}} + \mathop{\mathrm{D}}_{j=1}^{\infty} \frac{1}{b_{k,k+j}}, \qquad k = 0, 1, \ldots, \tag{12}$$

belong to the domain

$$\{z \in \mathbb{C} : \operatorname{Re} z > 0, |\arg z| < \theta\}, \ \theta < \frac{\pi}{2},$$

and let the following conditions be satisfied:

$$\lim_{r\to\infty} (\cos\theta)^{-2r} A_r^1 = 0, \qquad \lim_{r\to\infty} (\cos\theta)^{-2r} A_r^2 = 0, \qquad \lim_{r\to\infty} (\cos\theta)^{-3r} A_r^3 = 0,$$

where

$$A_r^k = \prod_{\ell=1}^{r} (1 + \mu_{\ell+1}^k), \qquad k = 1,2,3,$$

$$\mu_\ell^1 = \sup_i \{|b_{i+\ell-1,i}| \operatorname{Re} b_{i+\ell,i}\}, \qquad \mu_\ell^2 = \sup_i \{|b_{i,i+\ell-1}| \operatorname{Re} b_{i,i+\ell}\},$$

$$\mu_\ell^3 = |b_{\ell-1,\ell-1}| \operatorname{Re} b_{\ell,\ell}, \qquad \ell = 2,3,\ldots, \qquad i = 0,1,\ldots.$$

Then the TDCF (12) is convergent and the following estimate is true:

$$|f_n - f_{4p+1}| \le \frac{1}{\operatorname{Re} b_{0,0}} (1 - \cos\theta)^{-1} (\cos\theta)^{-4p-2} (A_{2p+2}^1 + A_{2p+2}^2)$$

$$+ \frac{2}{\operatorname{Re} b_{0,0}} (1 - \cos\theta)^{-1} (\cos\theta)^{-3p-1} A_{p+1}^3 + \frac{1}{\operatorname{Re} b_{0,0}} (\cos\theta)^{-2p-1} A_{2p}^3,$$

where $n > 4p+1$.

A somewhat different structure of two-dimensional continued fractions was proposed by Siemaszko [86]. The BCF corresponding to series (7) can be constructed in the form of BCF with independent variables

$$b_0 + \mathop{\mathrm{D}}_{k=1}^{\infty} \sum_{i_k=1}^{i_{k-1}} \frac{a_{i(k)} z_{i_k}}{1}, \tag{13}$$

where $i_0 = N$. These fractions are now extensively investigated. In [3, 8–11, 16, 17, 29, 30, 44, 47, 74, 75], the authors considered various criteria of convergence for these fractions. At the same time, the relationship with multiple power series was established in [12, 41, 43, 45, 46, 74, 75].

For branched continued fractions of a special form, Baran [8, 9, 11, 12, 46] established circular domains of convergence, which are multidimensional generalizations of some known theorems on the twin convergence sets of continued fractions (Leighton, Wall, Thron, Lange, Wyshinski, and McLaughlin) [76, 83]. In the case where $N = 1$, under certain conditions imposed on the parameters, the circular sets of convergence obtained by Baran in [8] are broader than in the theorems mentioned above.

Let

$$I = \{i(k): i(k) = i_1 i_2 \ldots i_k,\ 1 \le i_p \le i_{p-1},\ p = 1,\ldots,k,\ k \ge 1\}$$

and let

$$\ell = \ell(i(k)) = \sum_{s=1}^{k} \delta_{i_k}^{i_s}.$$

We split the set of multiindices I into three mutually disjoint subsets

$$I_1 = \{i(k): \ell = 1, k \ge 1\}, \qquad I_2 = \{i(k): \ell = 2m, k \ge 2\},$$

$$I_3 = \{i(k): \ell = 2m+1, k \ge 2\}.$$

Theorem 6. *The branched continued fraction* $(N > 1)$

$$1 + \mathop{D}_{k=1}^{\infty} \sum_{i_k=1}^{i_{k-1}} \frac{c_{i(k)}^2}{1} \tag{14}$$

with complex elements $c_{i(k)}$ *converges if the following conditions are satisfied:*

$$\left|c_{i(k)} \pm i\Gamma_{1,i_k}\right| \le \xi_{1,i_k}, \qquad (\xi_{1,i_k} + \left|\Gamma_{1,i_k}\right|)^2 \le \frac{\rho_1 - \varepsilon_1}{i_{k-1} - 1}, \qquad i(k) \in I_1,$$

$$\left|c_{i(k)} \pm i\Gamma_{2,i_k}\right| \ge \xi_{2,i_k}, \qquad (\xi_{2,i_k} - \left|\Gamma_{2,i_k}\right|)^2 \ge (2+\rho_1)(1+\rho_1+\rho+\varepsilon_2), \qquad i(k) \in I_2,$$

$$\left|c_{i(k)} \pm i\Gamma_{3,i_k}\right| \le \xi_{3,i_k}, \qquad (\zeta_{3,i_k} + \left|\Gamma_{3,i_k}\right|)^2 \le \rho - \varepsilon_3, \qquad i(k) \in I_3,$$

where $\rho_1 > 0$, $\rho > 0$, $0 < \varepsilon_1 < \rho$, $0 < \varepsilon_3 < \rho$, $\varepsilon_2 > 0$, $\Gamma_{s,i_k} \in \mathbb{C}$, $\xi_{s,i_k} > 0$, *and* $s = 1,2,3$.

For the TDCF and BCF of a special form, Kuchmins'ka proposed boundary versions of the Worpitzky theorem [52, 82]. We present one of these versions:

Theorem 7. *Let* ρ *be a real number from* $(0,S]$ *and let* F_ρ *be a family of two-dimensional continued fractions (10) with partial denominators equal to 1 whose elements satisfy the following conditions:*

$$\left|a_{i+1,i}\right| + \left|a_{i,i+1}\right| + \left|a_{i+1,i+1}\right| = \rho(1-\rho), \qquad \left|a_{0,0}\right| = \rho(1-\rho),$$

$$\left|a_{i+j,i}\right| = \rho(1-\rho), \qquad \left|a_{i,i+j}\right| = \rho(1-\rho), \qquad j \ge 2, \qquad 0 < \rho \le \frac{1}{2}.$$

Then the sets of all possible values of f *for two-dimensional continued fractions (11) from* F_ρ *form a ring*

$$\rho\frac{(1-\rho)}{1+\rho} \le |f| \le \rho.$$

Dmytryshyn studied some classes of functional branched continued fractions with independent variables, established the conditions of their convergence, determined estimates for the errors of approximation by approximants and proposed various multidimensional algorithms for the expansion of multiple power series in these fractions including, in particular, the Bauer g-algorithm, the Rutishauser qd-algorithm, etc. [42–47, 74, 75]. He constructed and investigated the expansions of some specific analytic functions in functional BCF with nonequivalent variables. Thus, the function

$$F(a,1,c;z_1)\,F(b,1,d;-z_2\,(F(a,1,c;-z_1))^2) = \sum_{\ell=0}^{\infty}(-1)^{\ell}\left(\sum_{k=0}^{\infty}(-1)^k\frac{(a)_k}{(c)_k}z_1^k\right)^{2\ell+1}\frac{(b)_\ell}{(d)_\ell}z_2^\ell$$

is expanded in a g-fraction with nonequivalent variables as follows:

$$\left(\Phi_0(z_1)+\mathop{\mathrm{D}}_{n=1}^{\infty}\frac{g_{0n}(1-g_{0n})z_2}{\Phi_n(z_1)}\right)^{-1},$$

$$\Phi_n(z_1) = 1+g_{1n}z_1\left(1+\mathop{\mathrm{D}}_{k=2}^{\infty}\frac{g_{kn}(1-g_{k-1,n})z_1}{1}\right)^{-1},$$

$$g_{2r-1,\ell-1} = \frac{a+r-1}{c+2r-2}, \qquad g_{2r,\ell-1} = \frac{r}{c+2r-1},$$

$$g_{0,2\ell-1} = \frac{b+\ell-1}{d+2\ell-2}, \qquad g_{0,2\ell} = \frac{\ell}{d+2\ell-1}.$$

In different classes of BCF, we consider various types of functional BCF, namely, the multidimensional g-, J-, π-, and C-fractions. The multidimensional g-fractions are studied especially comprehensively. The results of their investigation were summarized in the survey [24]. Hoyenko [37] studied the relationship between the correspondence and uniform convergence of functional BCF.

Bubnyak defined and established, by using the properties of boundary-periodic and inverse continued fractions, the criteria of pointwise and uniform convergence of periodic BCF of a special form and, in particular, investigated the oval domains of convergence for p-periodic BCF [16, 17, 29, 30, 73]. We now present the necessary condition of convergence of a 1-periodic BCF with real elements formulated in the following theorem [28, p. 99]:

Theorem 8. *If a 1-periodic BCF*

$$\left(1+\mathop{\mathrm{D}}_{k=1}^{\infty}\sum_{i_k=1}^{i_{k-1}}\frac{c_{i_k}}{1}\right)^{-1} \tag{15}$$

with real elements c_{i_k} *converges, then its elements satisfy the conditions*

$$c_q \geq -\frac{1}{4} X_{q-1}^2, \quad q = 1, 2, \ldots, N,$$

where X_q *are determined by the recurrence relation*

$$X_q = \frac{1}{2}\left(X_{q-1} + \sqrt{X_{q-1}^2 + 4c_q}\right), \quad X_0 = 1.$$

The sufficient condition of convergence of the BCF (15) can be formulated as follows:

$$c_q > -\frac{1}{4} X_{q-1}^2, \quad q=1, 2, \ldots, N.$$

Note that the presented list of references is far from being complete.

REFERENCES

1. T. M. Antonova, "Multidimensional generalization of the theorem on parabolic domains of convergence of continued fractions," *Mat. Met. Fiz.-Mekh. Polya,* **42,** No. 4, 7–12 (1999).
2. T. M. Antonova, "Convergence of branched continued fractions with complex elements and of their even parts," *Mat. Met. Fiz.-Mekh. Polya,* **46,** No. 4, 7–15 (2003).
3. T. M. Antonova and D. I. Bodnar, "Domains of convergence of branched continued fractions of a special form," in: *Approximation Theory of Functions and Its Application: Proceedings of the Institute of Mathematics* [in Ukrainian], Vol. 31, Institute of Mathematics, Ukrainian National Academy of Sciences (2000), pp. 19–32.
4. T. M. Antonova and V. R. Hladun, "Sufficient conditions of convergence and stability of branched continued fractions with alternating partial numerators," *Mat. Met. Fiz.-Mekh. Polya,* **47,** No. 4, 27–35 (2004).
5. T. M. Antonova and O. M. Sus', "Sufficient conditions of the equivalent convergence for sequences of different approximations of two-dimensional continued fractions," *Mat. Met. Fiz.-Mekh. Polya,* **58,** No. 4, 7–14 (2015).
6. T. M. Antonova and O. M. Sus,' "Necessary conditions of convergence for one class of two-dimensional continued fractions with complex elements," in: *Theory of Approximation of Functions and Related Problems: Proceedings of the Institute of Mathematics* [in Ukrainian], Vol. 12, No. 4, Institute of Mathematics, Ukrainian National Academy of Sciences (2015), pp. 8–28.
7. T. M. Antonova and O. M. Sus,' "On twin convergence sets for two-dimensional continued fractions with complex elements," *Mat. Met. Fiz.-Mekh. Polya,* **50,** No. 3, 94–101 (2007).
8. O. E. Baran, "Some circular regions of convergence for branched continued fractions of a special form," *Mat. Met. Fiz.-Mekh. Polya,* **56,** No. 3, 7–14 (2013); ***English translation:*** *J. Math. Sci.,* **205**, No. 4, 491–500 (2015).
9. O. E. Baran, "Some domains of convergence of branched continued fractions of a special form," *Karpat. Mat. Publ.,* **5**, No. 1, 4–13 (2013).
10. O. E. Baran, *Approximation of Functions of Several Variables by Branched Continued Fractions with Nonequivalent Variables* [in Ukrainian], Candidate-Degree Thesis (Physics and Mathematics), Ivano-Frankivs'k (2014).
11. O. E. Baran, "Twin circular domains of convergence of branched continued fractions with nonequivalent variables," *Mat. Met. Fiz.-Mekh. Polya,* **52,** No. 4, 73–80 (2009); ***English translation****, Math. Sci.,* **174**, No. 2, 209–218 (2011).
12. O. E. Baran and D. I. Bodnar, "Expansion of a multiple power series in a multidimensional C-fraction with nonequivalent variables," *Volyn. Mat. Visn.,* Issue 6, 15–20 (1999).
13. D. I. Bodnar, *Branched Continued Fractions* [in Russian], Naukova Dumka, Kiev, (1986).
14. D. I. Bodnar, "Analytic theory of branched continued fractions: history, main results, and unsolved problems," *Mat. Met. Fiz.-Mekh. Polya,* **50,** No. 3, 21–29 (2007).
15. D. I. Bodnar, "On the convergence of branched continued fractions," *Mat. Met. Fiz.-Mekh. Polya,* **41,** No. 1, 117–126 (1998); ***English translation****: J. Math. Sci.,* **97**, No. 1, 3862–3871 (1999).
16. D. I. Bodnar and M. M. Bubnyak, "Multidimensional generalization of the oval theorem for a periodic branched continued fraction of a special form," in: *Mathematical Problems of Mechanics and Computational Mathematics: Proceedings of the Institute of Mathematics* [in Ukrainian], Vol. 11, No. 4, Institute of Mathematics, Ukrainian National Academy of Sciences (2014), pp. 54–67.

17. D. I. Bodnar and M. M. Bubnyak, "Estimates of the rate of pointwise and uniform convergence for one-periodic branched continued fractions of a special form," *Mat. Met. Fiz.-Mekh. Polya,* **56,** No. 4, 24–32 (2013); ***English translation****: J. Math. Sci.,* **208**, No. 3, 289–300 (2015).
18. D. I. Bodnar, O. H. Voznyak, and R. I. Mykhal'chuk, "A criterion of convergence of a branched continued fraction with positive elements," *Mat. Met. Fiz.-Mekh. Polya,* **58,** No. 1, 57–64 (2015); ***English translation:*** *J. Math. Sci.,* **222**, No. 1, 70–80 (2017).
19. D. I. Bodnar and V. R. Hladun, "Some domains of stability under perturbations for branched continued fractions with complex elements," *Nauk. Visn. Cherniv. Univ. Matematyka,* Issue 288, 18–27 (2006).
20. D. I. Bodnar and V. R. Hladun, "Sufficient conditions of stability of branched continued fractions with positive elements," *Mat. Met. Fiz.-Mekh. Polya,* **45,** No. 1, 22–27 (2002).
21. D. I. Bodnar and V. R. Hladun, "On the stability under perturbations for branched continued fractions with complex elements," *Mat. Studii,* **25**, No. 2, 207–212 (2006).
22. D. I. Bodnar and N. P. Hoyenko, "Approximation of the ratio of Lauricella functions by a branched continued fraction," *Mat. Studii,* **20**, No. 2, 210–214 (2003).
23. D. I. Bodnar and N. P. Hoyenko, "Approximation of Lauricella hypergeometric functions by multidimensional generalizations of Nörlund continued fractions," in: *Proc. of the Ukrainian Mathematical Congress-2001. Section 10. Theory of Approximations and Harmonic Analysis* [in Ukrainian], Institute of Mathematics, Ukrainian National Academy of Sciences, Kyiv (2002), pp. 34–44.
24. D. I. Bodnar and Kh. Yo. Kuchmins'ka, "Multidimensional generalizations of g-fractions," *Mat. Met. Fiz.-Mekh. Polya,* **51,** No. 2, 34–41 (2008); ***English translation:*** *J. Math. Sci.,* **162**, No. 1, 34–43 (2009).
25. D. I. Bodnar and Kh. I. Kuchmins'ka, "Branched continued fractions (30th anniversary of the first publication)," *Mat. Met. Fiz.-Mekh. Polya,* **39,** No. 2, 9–19 (1996); ***English translation:*** *J. Math. Sci.,* **90**, No. 5, 2324–2333 (1998).
26. D. I. Bodnar and O. S. Manzii, "Expansion of the ratio of Appel hypergeometric functions F_3 into a branching continued fraction and its limit behavior," *Mat. Met. Fiz.-Mekh. Polya,* **41,** No. 4, 12–16 (1998); ***English translation:*** *J. Math. Sci.,* **107**, No. 1, 3550–3554 (2001).
27. P. I. Bodnarchuk and V. Ya. Skorobohat'ko, *Branched Continued Fractions and Their Applications* [in Ukrainian], Naukova Dumka, Kyiv (1974).
28. M. M. Bubnyak, *Sets of Convergence for Periodic Branched Continued Fractions of a Special Form* [in Ukrainian], Candidate-Degree Thesis (Physics and Mathematics), Ivano-Frankivs'k (2016).
29. M. M. Bubnyak, "Criteria of convergence periodic branched continued fractions of a special form," in: *Approximation Theory of Functions and Related Problems: Proceedings of the Institute of Mathematics* [in Ukrainian], Vol. 12, No. 4, Institute of Mathematics, Ukrainian National Academy of Sciences (2015), pp. 54–66.
30. M. M. Bubnyak, "Estimates of the rate of convergence of a 1-periodic branched continued fraction of a special form," *Karpat. Mat. Publ.,* **5**, No. 2, 187–195 (2013).
31. S. M. Vozna, "Convergence of a two-dimensional continued g-fraction," *Mat. Met. Fiz.-Mekh. Polya,* **47,** No. 3, 28–32 (2004).
32. S. M. Vozna, *Approximation of Functions by Continued and Two-Dimensional Continued Fractions* [in Ukrainian], Candidate-Degree Thesis (Physics and Mathematics), Lviv (2008).
33. V. R. Hladun, *Analysis of Stability under Perturbations of Branched Continued Fractions* [in Ukrainian], Candidate-Degree Thesis (Physics and Mathematics), Lviv (2007).
34. V. R. Hladun, "Convergence criteria and stability of branched continued fractions with negative partial numerators," *Mat. Met. Fiz.-Mekh. Polya,* **46,** No. 4, 16–26 (2003).
35. N. Hoyenko, "Convergence of the expansion of the ratio of Lauricella F_D functions into a branched continued fraction," *Mat. Met. Fiz.-Mekh. Polya,* **45,** No. 4, 61–66 (2002).
36. N. Hoyenko, *Approximation of Lauricella Hypergeometric Functions by Branched Continued Fractions* [in Ukrainian], Candidate-Degree Thesis (Physics and Mathematics), Lviv (2004).
37. N. P. Hoyenko, "On the convergence of expansion of the ratio of Lauricella hypergeometric functions into branched continued fractions," in: *Theory of Approximation of Functions and Its Applications: in: Mathematical Problems of Mechanics and Computational Mathematics: Proceedings of the Institute of Mathematics* [in Ukrainian], Vol. 31, Institute of Mathematics, Ukrainian National Academy of Sciences (2000), pp. 135–143.
38. N. Hoyenko, "Conformity principle and the convergence of sequences of analytic functions of several variables," *Mat. Visn. Nauk. Tovar. Shevchenka,* **4,** 42–48 (2007).
39. N. Hoyenko, T. Antonova, and S. Rakintsev, "Approximation of the ratio of Lauricella–Saran functions F_S with real parameters by branched continued fractions," *Mat. Visn. Nauk. Tovar. Shevchenka,* **8**, 28–42 (2011).
40. I. I. Demkiv, "Interpolation integral continued fractions for functionals of two variables," *Visn. Shevchenko Kyiv. Nats. Univ., Ser. Fiz.-Mat. Nauky,* No. 3, 143–148 (2011).
41. R. I. Dmytryshyn, "Multidimensional generalization of the Bauer g-algorithm," *Karpat. Mat. Publ.,* **4**, No. 2, 247–260 (2012).
42. R. I. Dmytryshyn, *Multidimensional Analogs of g-Fractions, Their Properties, and Criteria of Convergence* [in Ukrainian], Candidate-Degree Thesis (Physics and Mathematics), Lviv (1998).
43. R. I. Dmytryshyn, "Associated branched continued fractions with two independent variables," *Ukr. Mat. Zh.,* **66**, No. 9, 1175–1184 (2014); ***English translation:*** *Ukr. Math. J.,* **66**, No. 9, 1312–1323 (2014).

44. R. I. Dmytryshyn, "On the convergence of a multidimensional g-fraction with independent variables," *Mat. Met. Fiz.-Mekh. Polya,* **48,** No. 4, 121–127 (2005).
45. R. I. Dmytryshyn, "On the expansion of some functions in a two-dimensional g-fraction with independent variables," *Mat. Met. Fiz.-Mekh. Polya,* **53,** No. 4, 28–34 (2010); ***English translation:*** *J. Math. Sci,* **181**, No. 3. 320–327 (2012).
46. R. I. Dmytryshyn and O. E. Baran, "Some types of branched continued fractions corresponding to multiple power series," in: *Theory of Approximation of Functions and Its Application: Proceedings of the Institute of Mathematics* [in Ukrainian], Vol. 31, Institute of Mathematics, Ukrainian National Academy of Sciences, Kyiv (2000), pp. 82–92.
47. R. Dmytryshyn, "On some domains of convergence of a multidimensional J-fraction with independent variables," *Mat. Visn. Nauk. Tovar. Shevchenka,* **8**, 69–77 (2011).
48. N. S. Dronyuk, "Expansion of some functions in branched continued fractions," in: *Proc. of the 2nd Sci. Conf. of Young Mathematicians of Ukraine* [in Ukrainian], Naukova Dumka, Kyiv (1966), pp. 185–189.
49. Kh. Yo. Kuchmins'ka, "Corresponding and associated branched continued fractions for a double power series," *Dop. Akad. Nauk Ukr. RSR. Ser. A,* No. 7, 613–617 (1978).
50. Kh. Yo. Kuchmins'ka, *Two-Dimensional Continued Fractions* [in Ukrainian], Pidstryhach Institute for Applied Problems in Mechanics and Mathematics, Ukrainian National Academy of Sciences, Lviv (2010).
51. Kh. Yo. Kuchmins'ka, "Two-dimensional regular C-fractions," *Mat. Met. Fiz.-Mekh. Polya,* **55,** No. 2, 7–15 (2012); ***English translation:*** *J. Math. Sci.,* **192**, No. 5, 485–497 (2013).
52. Kh. Yo. Kuchmins'ka, "Boundary versions of the Worpitzky theorem for two-dimensional continued fractions," *Ukr. Mat. Zh.,* **66**, No. 8, 1106–1116 (2014); ***English translation:*** *Ukr. Math. J.,* **66**, No. 8, 1236–1247 (2014).
53. Kh. Yo. Kuchmins'ka, "On the properties of two-dimensional continued fractions," in: *Theory of Approximation of Functions and Related Problems: Proceedings of the Institute of Mathematics* [in Ukrainian], Vol. 7, No. 1, Institute of Mathematics, Ukrainian National Academy of Sciences (2010), pp. 113–127.
54. Kh. Yo. Kuchmins'ka, "On sufficient conditions of convergence of two-dimensional continued fractions," *Dop. Nats. Akad. Nauk Ukr.,* No. 12, 11–16 (2000).
55. Kh. Yo. Kuchmins'ka, "On the convergence of two-dimensional continued fractions," in: *Theory of Approximation of Functions and its Applications: Proceedings of the Institute of Mathematics* [in Ukrainian], Vol. 31, Institute of Mathematics, Ukrainian National Academy of Sciences (2000), pp. 282–296.
56. Kh. Yo. Kuchmins'ka, "On Worpitzky-like theorems for a two-dimensional continued fraction," *Mat. Met. Fiz.-Mekh. Polya,* **53,** No. 3, 17–26 (2010); ***English translation:*** *J. Math. Sci.,* **180**, No. 1, 1–14 (2012).
57. Kh. Yo. Kuchmins'ka, *Development of the Analytic Theory of Two-Dimensional Continued Fractions* [in Ukrainian], Doctoral-Degree Thesis (Physics and Mathematics), Lviv (2012).
58. Kh. Yo. Kuchmins'ka, "Stability in the calculation of two-dimensional continued fractions," *Mat. Met. Fiz.-Mekh. Polya,* **56,** No. 4, 15–23 (2013); ***English translation***: *J. Math. Sci.,* **208**, No. 3, 277–288 (2015).
59. Kh. Yo. Kuchmins'ka, O. M. Sus', and S. M. Vozna, "Approximation properties of two-dimensional continued fractions," *Ukr. Mat. Zh.,* **55**, No. 1, 30–44 (2003); ***English translation***: *Ukr. Math. J.,* **55**, No. 1, 36–54 (2003).
60. V. L. Makarov, V. V. Khlobystov, and B. R. Mykhal'chuk, "Interpolational integral continued fractions," *Ukr. Mat. Zh.,* **55**, No. 4, 479–488 (2003); ***English translation***: *Ukr. Math. J.,* **55**, No. 4, 576–587 (2003).
61. O. S. Manzii, *Approximation of the Appel Hypergeometric Functions by Branched Continued Fractions* [in Ukrainian], Candidate-Degree Thesis (Physics and Mathematics), Lviv (2000).
62. O. S. Manzii, "On the convergence of expansion of the ratio of Appel hypergeometric functions F_3 in a branched continued fraction in an infinite domain," *Mat. Met. Fiz.-Mekh. Polya,* **42**, No. 2, 7–11 (1999).
63. O. S. Manzii, "Expansion of the ratio of Appel hypergeometric functions F_3 in a branched continued fraction," *Visn. Derzh. Univ. "L'vivs'ka Politekhnika,* No. 346, 3–9 (1998).
64. M. O. Nedashkovs'kyi, "Criteria of convergence of matrix branched continued fractions," *Mat. Met. Fiz.-Mekh. Polya,* **46,** No. 4, 50–56 (2003).
65. M. O. Nedashkovs'kyi, "Solution of nonlinearly polynomial equations by using branched continued fractions," *Komp'yuting,* **2**, Issue 1 (No. 3), 83–87 (2003).
66. M. O. Nedashkovs'kyi and O. Ya. Koval'chuk, *Calculations with λ-Matrices* [in Ukrainian], Naukova Dumka, Kyiv (2007).
67. M. M. Pahirya, *Approximation of Functions by Continued Fractions* [in Ukrainian], Grazhda, Uzhgorod (2016).
68. M. M. Pahirya and R. A. Katsala, "Properties of reciprocal derivatives," *Ukr. Mat. Zh.,* **62**, No. 5, 708–713 (2010); ***English translation:*** *Ukr. Math. J.,* **62**, No. 5, 816–823 (2010).
69. V. Ya. Skorobohat'ko, *Theory of Branched Continued Fractions and Its Application in Computational Mathematics* [in Russian], Nauka, Moscow (1983).
70. V. Ya. Skorobohat'ko, N. S. Dronyuk, O. I. Bobyk, and B. Yo. Ptashnyk, "Branched continued fractions and their applications," in: *Proc. of the 2nd Scientific Conf. of Young Mathematicians of Ukraine* [in Ukrainian], Naukova Dumka, Kyiv (1966), pp. 561–565.
71. O. M. Sus', "On the evaluation of the rate of convergence of two-dimensional continued fractions with complex elements," *Prykl. Probl. Mekh. Mat.,* Issue 6, 115–123 (2008).
72. M. S. Syavavko, *Integral Continued Fractions* [in Ukrainian], Naukova Dumka, Kyiv (1994).

73. D. I. Bodnar and M. M. Bubniak, "On convergence of (2,1,...,1)-periodic branched continued fraction of the special form," *Karpat. Mat. Publ.,* **7**, No. 2, 148–154 (2015).
74. D. I. Bodnar and R. I. Dmytryshyn, "On the convergence of multidimensional g-fraction," *Mat. Studii,* **15**, No. 2, 115–126 (2001).
75. R. I. Dmytryshyn, "The two-dimensional g-fraction with independent variables for double power series," *J. Approx. Theory,* **164,** No. 12, 1520–1539 (2012).
76. W. B. Jones and W. J. Thron, "Continued fractions: analytic theory and applications," in: *Encyclopedia of Mathematics and Its Applications,* Vol. 11, Addison-Wesley, Reading, MA (1980).
77. Kh. Kuchmins'ka, "On sufficient conditions for convergence of two-dimensional continued fractions" *Acta Applicandae Math.,* **61**, No. 1-3, 175–183 (2000).
78. Kh. Kuchmins'ka, "On the "fork" property for two-dimensional continued fractions," *Comm. Analyt. Theory Continued Fractions,* **7**, 32–35 (1997).
79. Kh. Kuchmins'ka, "Some properties of two-dimensional continued fractions," *J. Comput. Appl. Math.,* **105**, No. 1-2, 347–353 (1999).
80. Kh. Yo. Kuchmins'ka and S. M. Vozna, "Truncation error bounds for a two-dimensional continued g-fraction," *Mat. Studii,* **24**, No. 2, 120–126 (2005).
81. Kh. Kuchminska, "Two-dimensional generalization of continued fractions," in: M. Katsurada, T. Komatsu, and H. Nakada (editors), *Diophantine Analysis and Related Fields-2006,* Dept. Math. Keio Univ. Seminar on Math. Sci., No. 35 (2006), pp. 125–140.
82. Kh. Yo. Kuchminska, "A Worpitzky boundary theorem for branched continued fractions of the special form," *Karpat. Mat. Publ.,* **8**, No. 2, 272–278 (2016).
83. L. Lorentzen and H. Waadeland, *Continued Fractions,* Vol. 1: *Convergence Theory,* Atlantis Press–Word Scientific, Amsterdam–Paris, (2008).
84. J. A. Murphy and M. R. O'Donohoe, "A two-variable generalization of the Stieltjes-type continued fractions," *J. Comput. Appl. Math.,* **4**, No. 3, 181–190 (1978).
85. O. Perron, *Die Lehre von den Kettenbrüchen,* Bd. 2: *Analytisch-Funktionentheoretische Kettenbrüche,* Teubner, Stuttgart (1957).
86. W. Siemaszko, "Branched continued fractions for double power series," *J. Comput. Appl. Math.,* **6**, No. 2, 121–125 (1980).
87. H. S. Wall, *Analytic Theory of Continued Fractions,* Van Nostrand, New York (1948).